The Quantum Allan Variance

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Allan variance

$$\sigma^2(\tau) = \frac{1}{2}\left\langle \left[\frac{1}{\tau}\int_{\tau}^{2\tau} y\mathrm{d}t - \frac{1}{\tau}\int_{0}^{\tau} y\mathrm{d}t \right]^2 \right\rangle$$

τ – averaging time

K. Chabuda, I.L., R.D.D., "The Quantum Allan Variance", arXiv:1601.01685

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τ – averaging time

y – fractional frequency

$$y(t) = \frac{\nu_{LO}(t) - \nu_0}{\nu_0} = \frac{\omega(t)}{\omega_0}$$

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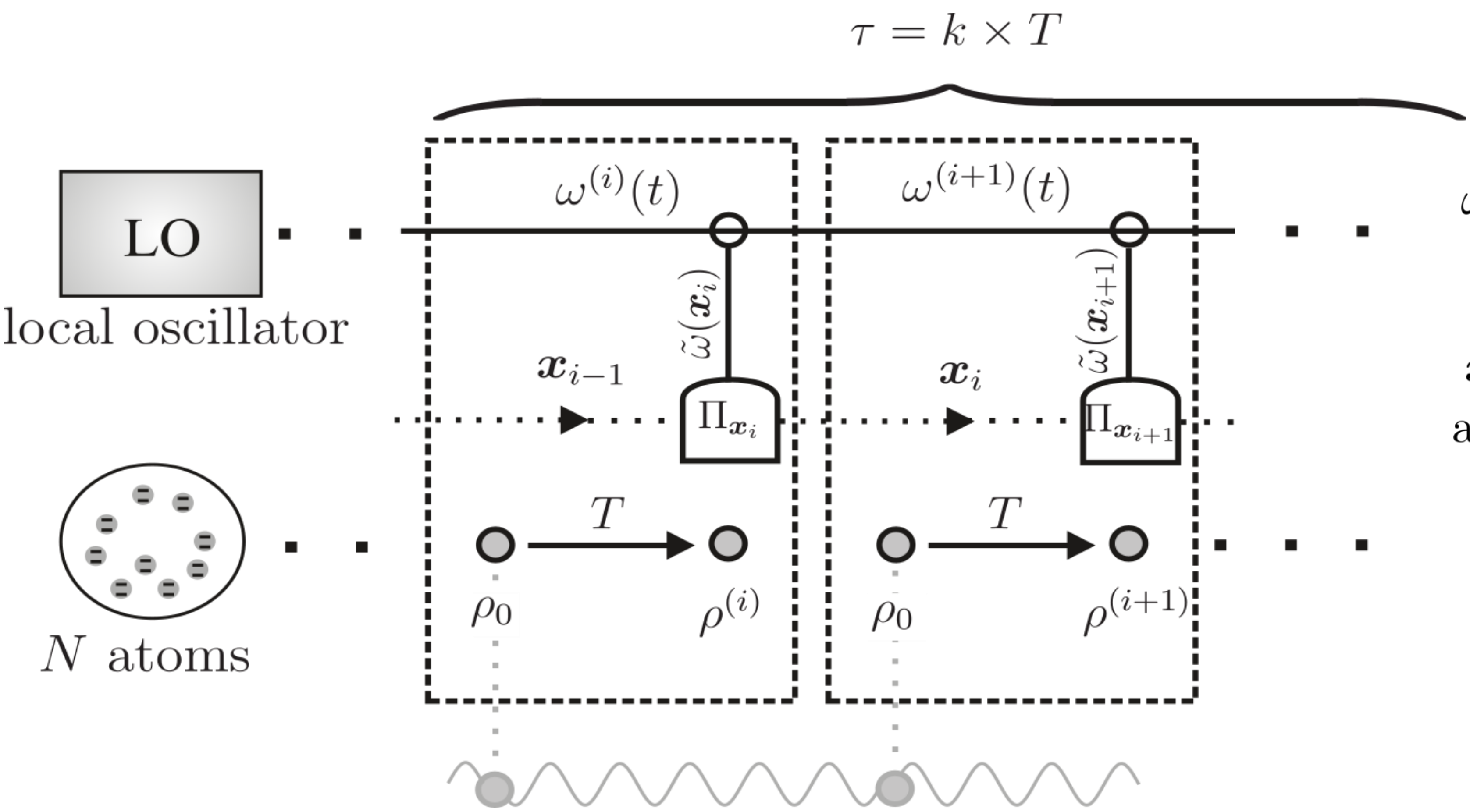
Scheme of atomic clock operation



$$\omega^{(i+1)}(t) = \omega_{\mathrm{LO}}(t) - \sum_{j=1}^{i} \tilde{\omega}(\boldsymbol{x}_j)$$

$$\boldsymbol{x}_{i-1} = (x_1 \ldots x_{i-1})$$

adaptiveness allowed

Looking for the fundamental bound

$$\sigma^2(\tau) = \frac{1}{2\omega_0^2}\left\langle \int \mathrm{d}\boldsymbol{x}_K p(\boldsymbol{x}_K)\left[\left(\frac{1}{\tau}\int_{\tau}^{2\tau}\mathrm{d}t\omega_{\mathrm{LO}}(t) - \frac{1}{\tau}\int_{0}^{\tau}\mathrm{d}t\omega_{\mathrm{LO}}(t)\right) - \tilde{\omega}^K(\boldsymbol{x}_K)\right]^2\right\rangle_{\omega_{\mathrm{LO}}}$$

$$K = 2k - 1$$

$$\tilde{\omega}^K(\boldsymbol{x}_K) = \frac{T}{\tau}\left(\sum_{i=1}^{k-1} i\tilde{\omega}(\boldsymbol{x}_i) + \sum_{i=k}^{K}(2k-i)\tilde{\omega}(\boldsymbol{x}_i)\right)$$

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$$\sigma^2(\tau) \geq \sigma_Q^2(\tau)$$

- Given: LO noise, atomic decoherence, number of atoms
- Find: optimal states, interrogation times, measurements and feedbacks (estimators) to minimize the Allan variance

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Optimal quantum Bayesian estimation strategy for the variance cost function

average cost : $\overline{\Delta^2\tilde{\omega}} = \int \mathrm{d}\omega p\left(\omega\right) \int \mathrm{d}x \mathrm{Tr}\left(\rho_\omega \Pi_x\right) \left(\tilde{\omega}\left(x\right) - \omega\right)^2 =$

$p\left(\omega\right)$ – prior distribution $\qquad p\left(x|\omega\right) = \mathrm{Tr}\left(\rho_\omega \Pi_x\right)$

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$p(\omega)$ – prior distribution $\qquad$ $p(x|\omega) = \mathrm{Tr}\left(\rho_\omega \Pi_x\right)$

$$= \int \mathrm{d}\omega p(\omega)\, \omega^2 + \mathrm{Tr}\left[\int \mathrm{d}\omega p(\omega)\, \rho_\omega \int \mathrm{d}x \tilde{\omega}^2(x)\, \Pi_x\right] - 2\mathrm{Tr}\left[\int \mathrm{d}\omega p(\omega)\, \omega \rho_\omega \int \mathrm{d}x \tilde{\omega}(x)\, \Pi_x\right]$$

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$$\bar{\rho} = \int \mathrm{d}\omega p(\omega)\,\rho_\omega \qquad \bar{\rho}' = \int \mathrm{d}\omega p(\omega)\,\omega\rho_\omega$$

$$L_2 = \int \mathrm{d}x \tilde{\omega}^2(x)\,\Pi_x \qquad L = \int \mathrm{d}x \tilde{\omega}(x)\,\Pi_x$$

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$$= \int \mathrm{d}\omega p(\omega)\, \omega^2 + \mathrm{Tr}\left[\int \mathrm{d}\omega p(\omega)\, \rho_\omega \int \mathrm{d}x \tilde{\omega}^2(x)\, \Pi_x\right] - 2\mathrm{Tr}\left[\int \mathrm{d}\omega p(\omega)\, \omega \rho_\omega \int \mathrm{d}x \tilde{\omega}(x)\, \Pi_x\right]$$

$$\bar{\rho} = \int \mathrm{d}\omega p(\omega)\, \rho_\omega \qquad \bar{\rho}' = \int \mathrm{d}\omega p(\omega)\, \omega \rho_\omega$$

$$L_2 = \int \mathrm{d}x \tilde{\omega}^2(x)\, \Pi_x \qquad L = \int \mathrm{d}x \tilde{\omega}(x)\, \Pi_x$$

Projective measurements are optimal* $\qquad \Pi_x^2 = \Pi_x \;\Rightarrow\; L_2 = L^2$

*K. Macieszczak, M. Fraas, R. Demkowicz – Dobrzanski, New J. Phys. **16**, 113002 (2014)

Optimal quantum Bayesian estimation strategy for the variance cost function

$$\overline{\Delta^2\tilde{\omega}} = \Delta^2\omega - \mathrm{Tr}\left[2\bar{\rho}' L - \bar{\rho} L^2\right]$$

$$\min_{\Pi_x, \tilde{\omega}(x)} \overline{\Delta^2\tilde{\omega}} \quad \Leftrightarrow \quad \min_{L} \overline{\Delta^2\tilde{\omega}} \qquad\qquad \frac{\mathrm{d}\overline{\Delta^2\tilde{\omega}}}{\mathrm{d}L} = 0 \quad \Rightarrow \quad \bar{\rho}' = \frac{1}{2}\left(\bar{\rho}L + L\bar{\rho}\right)$$

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$$\overline{\Delta^2\tilde{\omega}} = \Delta^2\omega - \mathrm{Tr}\left[\bar{\rho}L^2\right]$$

$$\bar{\rho} = \int \mathrm{d}\omega p\left(\omega\right)\rho_\omega \qquad\qquad \bar{\rho}' = \int \mathrm{d}\omega p\left(\omega\right)\omega\rho_\omega$$

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Quantum Allan Variance

$$\sigma^2(\tau) = \frac{1}{2\omega_0^2}\left\langle \int \mathrm{d}\boldsymbol{x}_K p(\boldsymbol{x}_K) \left[\left(\frac{1}{\tau}\int_{\tau}^{2\tau} \mathrm{d}t\omega_{\mathrm{LO}}(t) - \frac{1}{\tau}\int_{0}^{\tau} \mathrm{d}t\omega_{\mathrm{LO}}(t) \right) - \tilde{\omega}^K(\boldsymbol{x}_K) \right]^2 \right\rangle_{\omega_{\mathrm{LO}}}$$

$$\overline{\Delta^2\tilde{\omega}} = \int \mathrm{d}\omega p\left(\omega\right) \int \mathrm{d}x \mathrm{Tr}\left(\rho_\omega \Pi_x\right) \left[\omega - \tilde{\omega}\left(x\right)\right]^2$$

Quantum Allan Variance

$$\sigma^2(\tau) = \frac{1}{2\omega_0^2}\left\langle \int d\boldsymbol{x}_K p(\boldsymbol{x}_K)\left[\left(\frac{1}{\tau}\int_{\tau}^{2\tau} dt\omega_{\mathrm{LO}}(t) - \frac{1}{\tau}\int_0^{\tau} dt\omega_{\mathrm{LO}}(t)\right) - \tilde{\omega}^K(\boldsymbol{x}_K)\right]^2\right\rangle_{\omega_{\mathrm{LO}}}$$

$$\overline{\Delta^2\tilde{\omega}} = \int d\omega p(\omega) \int dx \mathrm{Tr}(\rho_\omega \Pi_x)[\omega - \tilde{\omega}(x)]^2$$

$$\sigma_Q^2(\tau) = \sigma_{\mathrm{LO}}^2(\tau) - \frac{1}{2\omega_0^2}\mathrm{Tr}\left[\bar{\rho}L^2\right]$$

$$\bar{\rho} = \left\langle \bigotimes_{i=1}^{K} U_{T,\mathrm{LO}}^{(i)}[\Lambda_T(\rho_0)]\right\rangle_{\omega_{\mathrm{LO}}}$$

$$U_{T,\mathrm{LO}}^{(i)} = \exp\left(-\mathrm{i}H\int_{(i-1)T}^{iT} dt\omega_{\mathrm{LO}}(t)\right)$$

$$\bar{\rho}' = \frac{1}{2}(\bar{\rho}L + L\bar{\rho})$$

$$\bar{\rho}' = \left\langle \int_0^{\tau} \frac{dt}{\tau}[\omega_{\mathrm{LO}}(t+\tau) - \omega_{\mathrm{LO}}(t)] \bigotimes_{i=1}^{K} U_{T,\mathrm{LO}}^{(i)}[\Lambda_T(\rho_0)]\right\rangle_{\omega_{\mathrm{LO}}}$$

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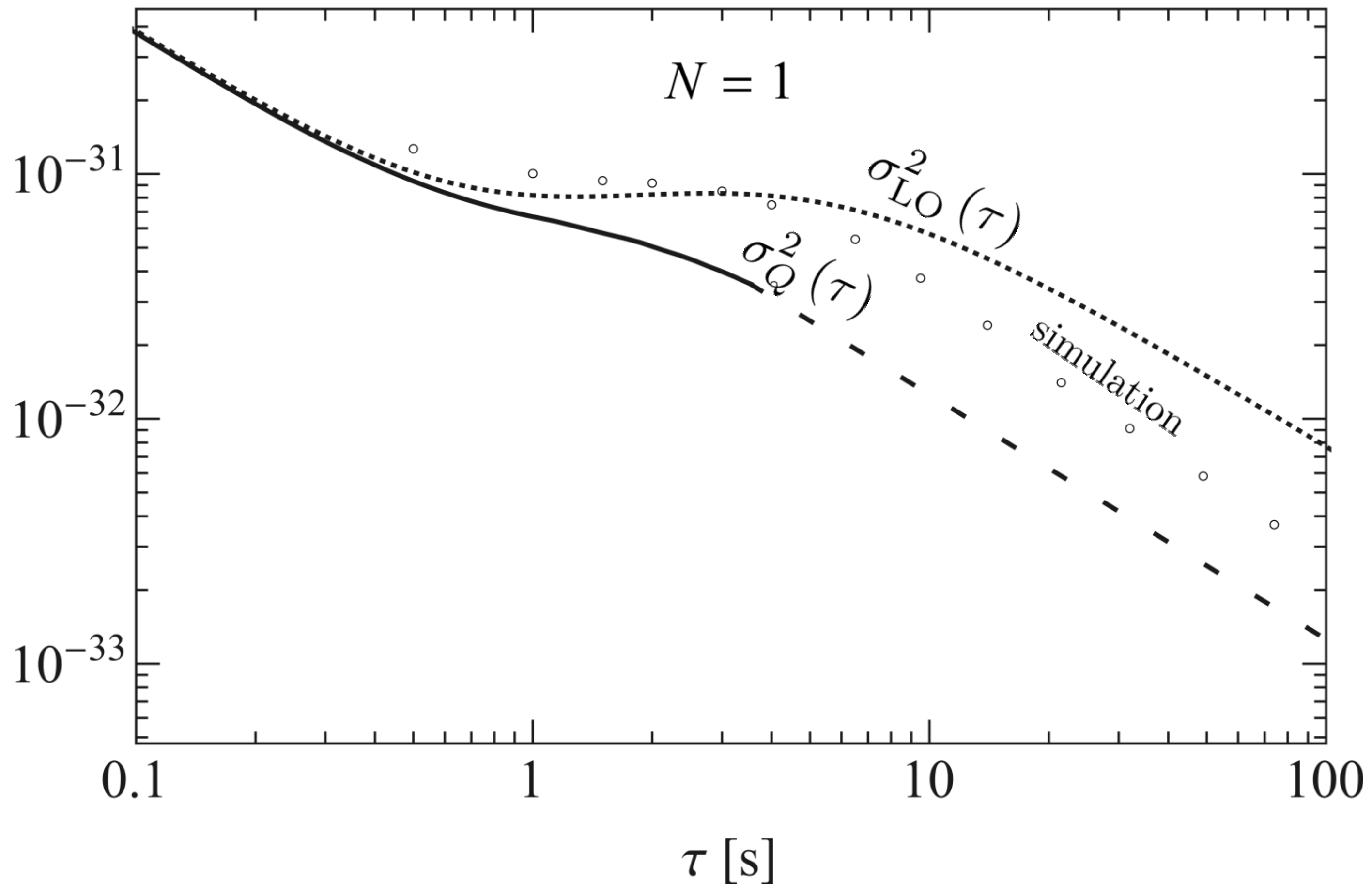
Results

autocorrelation function* :

$$R(t) = \langle \omega_{\mathrm{LO}}(t)\, \omega_{\mathrm{LO}}(0) \rangle_{\mathrm{LO}} = \alpha e^{-\gamma t} + \beta \delta(t)$$



$$\tau = k \times T$$

$$(N+1)^{2k+1}$$

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*Y. Y. Jiang et al., Nature Photon. **5**, 158 (2011)

Summary

- We provide an explicit method to find the ultimate bound on the Allan variance of an atomic clock in the most general scenario where N atoms are prepared in an arbitrarily entangled state and arbitrary measurement and feedback schemes are allowed.
- While our method is rigorous and completly general, it becomes numerically inefficient for large N and long averaging times. We think that this could be remedied by using matrix product state approximation.

Thank you for your attention!