Fundamental bounds on the clock time variance of atomic clocks

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http://arxiv.org/pdf/1303.6083.pdf

Abstract

1. Mathematical aspects and challenges in the atomic clock theory
2. Analytical bounds on long time stability of atomic clocks

How well we can measure time?

Classical mechanics: There is no obstacle to built a perfect clock within the theory.

Quantum mechanics: Perfect clock requires infinite mass and infinite power.

Atomic clock model: variables and notation

- ω_0 – Quantum reference frequency
- $\omega(t)$ – Clocks LO frequency; The relative frequency error is

$$y(t) = \frac{\omega(t) - \omega_0}{\omega_0}$$

- t_{clock} – Clock time;

$$t_{clock} = \frac{1}{\omega_0}\int_0^t \omega(s)\mathrm{d}s; \quad t_{clock} - t = \int_0^t y(s)\mathrm{d}s$$

- $y_{LO}(t)$ – LO frequency in absence of a feedback
- $\hat{y}$ – Feedback upon which

$$y(t) \quad \to \quad y(t) - \hat{y}$$

Atomic clock specifications

- Free running LO noise $y_{LO}(t)$
- Frequency reference; Initial state ρ; Interrogation dynamics;
- Feedback protocol

Local oscillator model

Principle mathematical properties:

- Markov property. The LO noise $y_{LO}(t)$ is a Markov process, $y_{LO}(t) = K(t,0)(y_{LO}(0))$.
- Martingale property. The LO frequency $\omega_{LO}(t)$ has no knowledge about ω_0 , e.g. $\mathbb{E}[y_{LO}(t)|y_{LO}(0)] = y_{LO}(0)$.

Adding two further reasonable properties, Autonomy and continuity we get

Example (Canonical example of LO-noise)

$$y_{LO}(t) = \sqrt{2D}W_t,$$

where W_t is the Wiener process.

Feedback model

Interrogation: In each time window $jT, (j+1)T$ the initial quantum state of the reference evolves as

$$\rho \quad \rightarrow \quad \rho(\bar{y}_j); \quad \bar{y}_j = \frac{1}{T}\int_{j-1T}^{jT} y(s)\mathrm{d}s.$$

Feedback: $\hat{y}_j$ is obtained by measurement $\Pi(\hat{y})$ on the state, i.e.

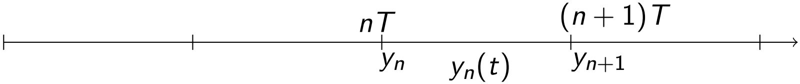
$$Prob(\hat{y}_j = \hat{y}) = \mathrm{Tr}(\rho(\bar{y}_j)\Pi(\hat{y})).$$

We denote $F_T(\varphi)$ the QFI of the family $\rho(\varphi)$ at a point φ.

Example (Hamiltonian model)

$$\rho(\bar{y}_j) = \exp(-iHT\bar{y}_j)\rho\exp(iHT\bar{y}_j), \quad F = 4T^2 < \Delta E^2 >_\rho$$

Full, closed loop model



Put $y_n(t) := y(t + nT), \quad t \in [0,\ T), \quad y_n := y_n(0).$

The evolution equations of the model are

$$y_n(t) = K_t y_n, \quad t \in [0, T)$$
$$y_{n+1} = K_T y_n - \hat{y}_n.$$

Given specifications $\rho(\varphi)$, $\Pi(\hat{y})$, K_t, T we determine $y(t)$.

Measures of instability

- Allan Variance: $\sigma^2(\tau) := \frac{1}{2}\mathbb{E}[\left(\frac{1}{\tau}\int_\tau^{2\tau} y(s)\mathrm{d}s - \frac{1}{\tau}\int_0^\tau y(s)\mathrm{d}s\right)^2]$

  For atomic clocks $\sigma^2(\tau) \sim D\tau^{-1}$ as $\tau \to \infty$.

- Clock Time Variance: $\mathbb{E}[(t_{clock} - t)^2]$

  For large t, $t^2\sigma^2(t) \sim \mathbb{E}[(t_{clock} - t)^2] \sim Dt$.

Unbiased clock

Unbiased clock is accurate in average, $\mathbb{E}[t_{clock}] = t$.

$$\Downarrow$$

$\mathbb{E}[y(s)] = 0$ provided $\mathbb{E}[y(0)] = 0$.

$$\Downarrow \text{ (variational argument)}$$

For some $\zeta \in \mathbb{R}$, $\mathbb{E}[y - \hat{y}|y] = \zeta y$; $(1 - \zeta)$ is the gain in the feedback loop.

Definition (ζ-unbiased clock)

The clock is ζ-unbiased if the estimation procedure is ζ-biased with $|\zeta| < 1$.

Stationary states - properties

Theorem

Suppose that $\{\rho_T(\varphi), \Pi(\hat{y}), K_T\}$ *is an* ζ*-unbiased clock and let* $F_T(\varphi)$ *be a Fisher inf. assoc. to the family* $\rho_T(\varphi)$*. Then for the associated clock time variance we have a bound*

$$\lim_{t\to\infty} \frac{\mathbb{E}[(t_{clock} - t)^2]}{t} \geq T\frac{1}{F_T} + T\sigma^2_{LO}(T)\frac{\beta}{(1-\zeta)^2},$$

where β *is a constant associated to the noise and*

$$\sigma^2_{LO}(T) := \mathbb{E}[(\bar{y}_n - y_n)^2].$$

Above $1/F_T$ *is a shorthand for* $\mathbb{E}[1/F_T(\bar{y}_n)]$*.*

Unitary model

For a unitary evolution $F_T = 4T^2(\Delta E)^2$, and a phenomenological dependence $\sigma^2_{LO}(T) = DT^\alpha$ we can find optimal T in the latter bound and we get

$$\lim_{t\to\infty} \frac{\mathbb{E}[(t_{clock} - t)^2]}{t} \geq C \left(\frac{1}{4\Delta^2 E}\right)^{\frac{\alpha+1}{\alpha+2}},$$

with constant

$$C = \frac{\alpha+2}{\alpha+1}\left(\frac{\frac{1}{2}(\alpha+2)D(\alpha+1)^2}{(1-\zeta)^2}\right)^{\frac{1}{\alpha+2}}.$$

$\alpha = 0$ is $1/f$ noise; $\alpha = 1$ is Wiener process.

Cramer-Rao bound

For a given prob. dist. $q(\varphi)$ we introduce an average QFI,

$$\tilde{F} = \int F(\varphi) \frac{\tilde{q}(\varphi)^2}{q(\varphi)} \mathrm{d}\varphi, \quad \tilde{q}(\varphi) = \frac{\int_\varphi^\infty s q(s) \mathrm{d}s}{\sigma(q)^2}.$$

Theorem

Let $\hat{\varphi}$ be an estimation of a ran. var. φ (of zero mean) with a prior prob. dist. $q(\varphi)$. Then it holds

$$\mathbb{E}[(\varphi - \hat{\varphi})^2] \geq \frac{1}{\tilde{F} + \mathbb{E}[\varphi^2]^{-1}}.$$

Mathematical Challenges

- Phase-Frequency ambiguity (entanglement)
- Asymptotic normality
- Mixing times
- Entropy production

Thank you for your attention!