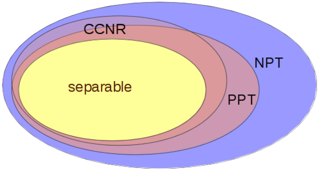
The Fisher Information Matrix as a tool in QIT

Otfried Gühne

S. Altenburg, G. Tóth, S. Wölk

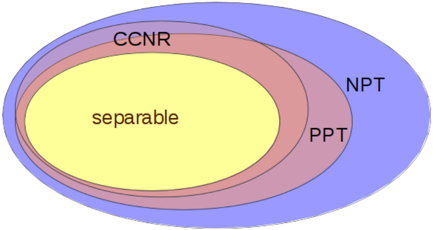


Department Physik, Universität Siegen

Overview

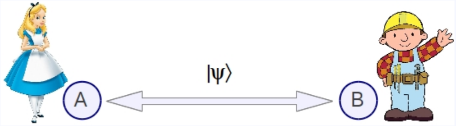
- Facts about entanglement
- Covariance matrices and entanglement
- The Fisher-Information and entanglement

Some facts about entanglement



What is entanglement?

Alice and Bob share a quantum state $|\psi\rangle$.



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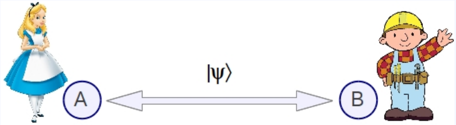

A pure state $|\psi\rangle$ is separable iff it is a product state:

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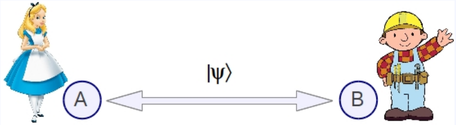

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Mixed states: Ask for convex combinations. ϱ is separable iff

$$\varrho = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \quad \text{mit} \quad p_i \geq 0, \ \sum_i p_i = 1.$$

Interpretation: Entanglement cannot be generated by local operations and classical communication.

R. Werner, PRA 40, 4277 (1989).

The separability problem

Open question

Given a state ϱ is it entangled or not?

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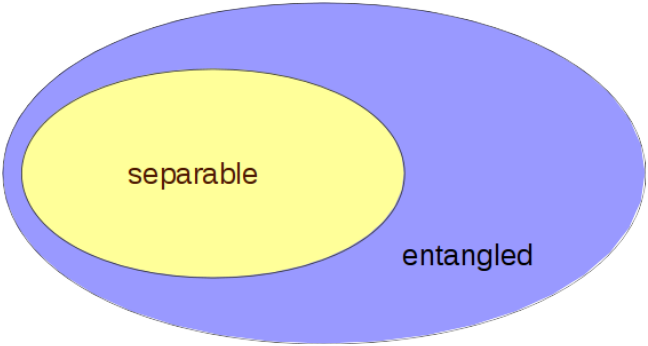
Geometrical picture

The set of separable states is a convex set.



The PPT criterion

Are there simple criteria to prove entanglement?

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Transposition and partial transposition

- Transposition: The usual transposition $X \mapsto X^T$ does not change the eigenvalues of the matrix X.
- For a product space on can consider the partial transposition: If $X = A \otimes B$:
$$X^{T_B} = A \otimes B^T$$

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Partial transposition and separability

Theorem. For a separable state, the partial transposition has no negative eigenvalues. ("ϱ is PPT" oder $\varrho^{T_B} \geq 0$).

Proof:

$$\varrho_{sep}^{T_B} = \sum_k p_k \varrho_A \otimes \varrho_B^T = \sum_k p_k \varrho_A \otimes \tilde{\varrho}_B \geq 0.$$

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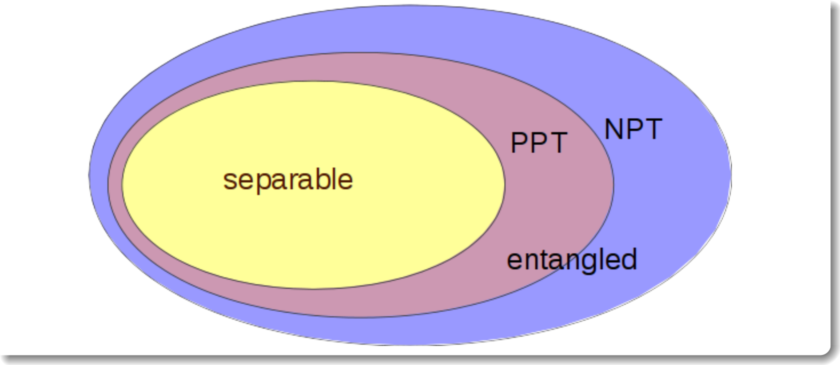
For two qubits: ϱ is PPT $\Leftrightarrow$ ϱ is separable. In general: $\not\Leftarrow$

A. Peres, PRL 77, 1413 (1996)

Geometrical view



The computable cross norm (CCNR) criterion

Schmidt decomposition (SD)

We write ϱ in the SD in operator space:

$$\varrho = \sum_k \lambda_k G_k^A \otimes G_k^B,$$

where $\lambda_k \geq 0$ and

$$Tr(G_k^A G_l^A) = Tr(G_k^B G_l^B) = \delta_{kl}.$$

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Oliver Rudolph, quant-ph/0202121.

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Example

The singlet can be written as

$$|\psi^-\rangle\langle\psi^-| = \frac{1}{2}\Big[\frac{\mathbb{1}}{\sqrt{2}} \otimes \frac{\mathbb{1}}{\sqrt{2}} - \frac{X}{\sqrt{2}} \otimes \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} \otimes \frac{Y}{\sqrt{2}} - \frac{Z}{\sqrt{2}} \otimes \frac{Z}{\sqrt{2}}\Big].$$

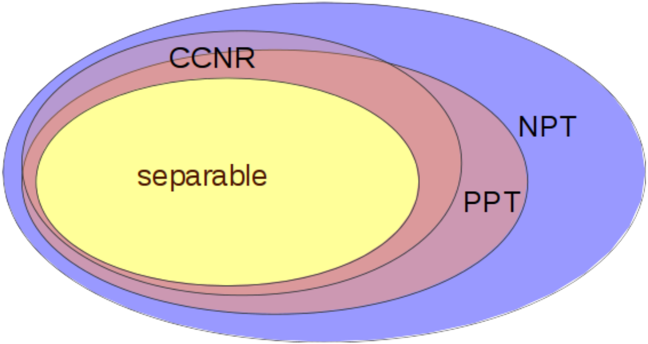
Hence $\sum_k \lambda_k = 2 \Rightarrow$ The state is entangled!

The CCNR criterion is also often called the realignment criterion and can detect bound entangled states.

K. Chen, L.-A. Wu, QIC 3, 193 (2003)

Geometrical view

The PPT criterion and the CCNR criterion are complementary.



Other criteria

There are *many* other separability criteria on the market:

- Reduction criterion: This is weaker than PPT.

$$\varrho \text{ is separable} \Rightarrow \mathbb{1} \otimes \varrho_A - \varrho \geq 0.$$

- Majorization criterion: This is weaker than PPT.

$$\varrho \text{ is separable} \Rightarrow \vec{\lambda}(\varrho_A) \succ \vec{\lambda}(\varrho).$$

- Other positive maps and symmetric extensions.

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- Other positive maps and symmetric extensions.
- Criteria using the Bloch representation of density matrices.

  J.I. de Vicente, QIC 7, 624 (2007).

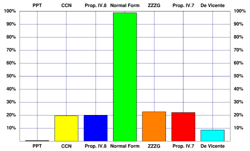
- Extensions of the CCNR criterion

  Zhang$^{\otimes 3}$ and Guo, PRA 76, 012334 (2007), PRA 77, 060301(R) (2008).

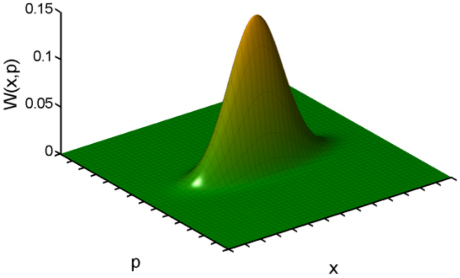
- Local uncertainty relations

The last three are interesting, as they are independent of PPT ...

Covariance matrices and entanglement



Covariance matrices in quantum optics



- Gaussian states are described by covariance matrices of X and P.
- Squeezed states are useful for quantum metrology (GEO600, 2018)

Local uncertainty relations (LURs)

LURs

Given observables A_i for Alice and B_i for Bob, with

$$\sum_k \Delta^2(A_k) \geq C_A, \quad \sum_k \Delta^2(B_k) \geq C_B$$

where $\Delta^2(A) = \langle A^2 \rangle - \langle A \rangle^2$ denotes the variance.
Then for separable states

$$\sum_k \Delta^2(A_k \otimes \mathbb{1} + \mathbb{1} \otimes B_k) \geq C_A + C_B$$

H. Hofmann, S. Takeuchi, PRA 68, 032103 (2003).

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Proof & Interpretation

- For $\varrho = \varrho_A \otimes \varrho_B$ we have $\Delta^2(A_k \otimes \mathbb{1} + \mathbb{1} \otimes B_k)_\varrho = \Delta^2(A_k)_{\varrho_A} + \Delta^2(B_k)_{\varrho_B}$.
- Separable states inherit the uncertainty relations from each party.

Example for the LURs

Take the A_i and B_i as Pauli matrices: $\{A_i\} = \{B_i\} = \{\sigma_x, \sigma_y, \sigma_z\} = \{X, Y, Z\}$. Then:

$$\sum_k \Delta^2(A_k) \geq 2, \quad \sum_k \Delta^2(B_k) \geq 2$$

Hence for separable states:

$$\sum_k \Delta^2(\sigma_k \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_k) \geq 4$$

But the singlet $|\psi^-\rangle$ is an eigenstate of $(\sigma_k \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_k)$.

$\Rightarrow$ The singlet is detected.

A surprising fact

The LUR derived above for the Pauli matrices requires for separable states

$$\mathcal{F}(\varrho) = \langle \mathbb{1} \otimes \mathbb{1} + X \otimes X + Y \otimes Y + Z \otimes Z \rangle - \frac{1}{2} \sum_{k=x,y,z} \langle \sigma_k \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_k \rangle^2 \geq 0.$$

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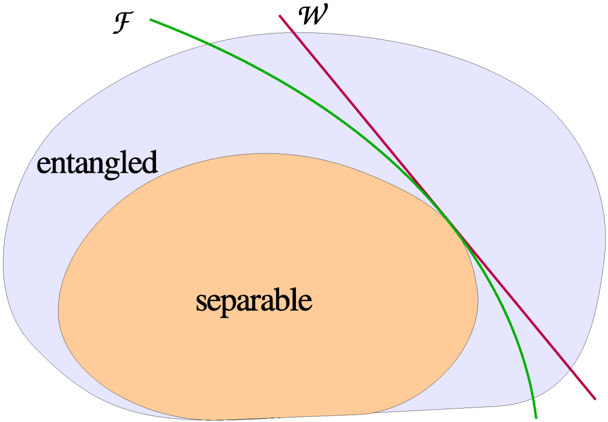
The linear part is known to be an *optimal* entanglement witness:

$$\mathcal{F}(\varrho) = 4 \cdot \langle \mathcal{W} \rangle - \frac{1}{2} \sum_{k=x,y,z} \langle \sigma_k \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_k \rangle^2 \geq 0$$

$\Rightarrow$ This special witness can be improved by a nonlinear witness!

Geometrical view

LURs can lead to nonlinear entanglement witnesses!



Covariance matrices: A systematic approach

Definition

Let M_k be some observables. The covariance matrix (CM) γ has the entries

$$\gamma_{ij} = \frac{\langle M_i M_j \rangle + \langle M_j M_i \rangle}{2} - \langle M_i \rangle \langle M_j \rangle.$$

γ is real, symmetric and positive semidefinite.

Note. Taking $M_1 = X$, $M_2 = P$ gives the well known covariance matrices for continuous variables / Gaussian states.

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Concavity property.

If $\varrho = \sum_k p_k \varrho_k$ then

$$\gamma(\varrho) \geq \sum_k p_k \gamma(\varrho_k).$$

Interpretation. Variances increase under mixing the states.

Covariance matrices

Bipartite systems.

For them we may take $\{M_k\} = \{A_k \otimes \mathbb{1}, \mathbb{1} \otimes B_k\}$ where A_k and B_k are a basis of the operator space (e.g. Pauli matrices for qubits). Then

$$\gamma = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix},$$

where $A = \gamma(\varrho_A, \{A_k\})$, $B = \gamma(\varrho_B, \{B_k\})$ and $C_{ij} = \langle A_i \otimes B_j \rangle - \langle A_i \rangle \langle B_j \rangle$.

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Product states.

If $\varrho = \varrho_A \otimes \varrho_B$ then $C = 0$, hence

$$\gamma(\varrho_A \otimes \varrho_B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

The covariance matrix criterion (CMC)

CMC

If $\varrho = \sum_k p_k |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k|$ is separable, then there exists

$$\kappa_A = \sum_k p_k \gamma(|a_k\rangle\langle a_k|) \text{ and } \kappa_B = \sum_k p_k \gamma(|b_k\rangle\langle b_k|)$$

such that

$$\gamma(\varrho) \geq \begin{bmatrix} \kappa_A & 0 \\ 0 & \kappa_B \end{bmatrix} =: \kappa_A \oplus \kappa_B.$$

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Remarks

- This is very similar to the criterion $\gamma \geq \gamma_A \oplus \gamma_B$ for Gaussian states.

  R. Werner, M. Wolf, PRL 86, 3658 (2001).

- How can we evaluate this criterion?
- For continuous variables we know that $\gamma_A \geq iJ$ but $\kappa_A \geq ??$

Tricks for the evaluation

One can use the following facts:

- One can prove that $Tr(\kappa_A) = d_A - 1$ (d_A is Alice's dimension).
- For 2×2 matrices we have:

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} \geq 0 \quad \Rightarrow \quad a + b \geq 2|c|$$

- More general:

$$\begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \geq 0 \quad \Rightarrow \quad Tr(X) + Tr(Y) \geq 2|Tr(Z)|$$

- Another trick: For any unitary invariant norm (e.g. trace norm)

$$\begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \geq 0 \quad \Rightarrow \quad \begin{bmatrix} \|X\| & \|Z\| \\ \|Z\| & \|Y\| \end{bmatrix} \geq 0$$

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K. Chen, L.-A. Wu, QIC 3, 193 (2003)

The CMC improves the CCNR

Idea. Take the observables from the SD for the CMC!

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Result

Define $g_k^A = Tr(G_k^A)$ and $g_k^B = Tr(G_k^B)$. Then for separable states

$$2\sum_k |\lambda_k - \lambda_k^2 g_k^A g_k^B| \leq 2 - \sum_k \lambda_k^2 \left[(g_k^A)^2 + (g_k^B)^2\right].$$

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Note. This looks ugly, but can be simply computed. Further, $\sum_k \lambda_k \leq 1$ follows from it.

$\Rightarrow$ The CMC is stronger than the CCNR criterion!

Local filters

Observation

Transformations of the type

$$\varrho \mapsto \tilde{\varrho} = (F_A \otimes F_B)\varrho(F_A \otimes F_B)^\dagger$$

preserve separability and entanglement. By this one can map ϱ to

$$\tilde{\varrho} = \frac{1}{d_A d_B}\left[\mathbb{1} + \sum_{i=1}^{d_A^2-1} \xi_i (\tilde{G}_i^A \otimes \tilde{G}_i^B)\right]$$

with *traceless* orthogonal observables $\tilde{G}_i^{A/B}$. The proof is constructive.

F. Verstraete, et al., PRA 68, 012103 (2003), J.M. Leinaas, et al., PRA 74, 012313 (2006).

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CMC under filtering

If ϱ in a $d \times d$ system is separable, then

$$\sum_{i=1}^{d^2-1} \xi_i \leq d^2 - d.$$

This criterion is necessary and sufficient for two qubits!

Further Results

- The CMC criterion is equivalent to the LURs.
- The CMC implies also other criteria besides CCNR:
  ZZZG, de Vicente ...

  J.I. de Vicente, QIC 7, 624 (2007), Zhang$^{\otimes 3}$, Guo, PRA 76, 012334 (2007), PRA 77, 060301(R) (2008).
- The CMC can be used to bound entanglement measures.

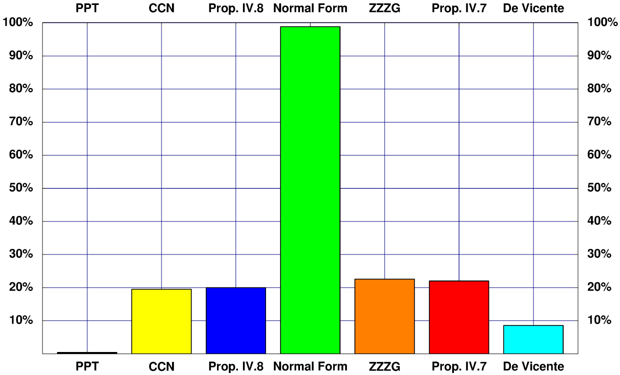
  O. Gittsovich et al, PRA 81, 032333 (2010)
- Open Question: The relation to the work on complementary of correlations.

  Talk by Lorenzo Maccone

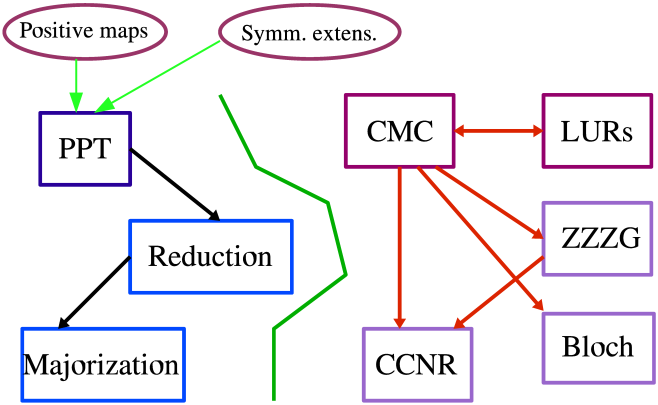
How strong are all these criteria?

Generate randomly chessboard states and check all the criteria.

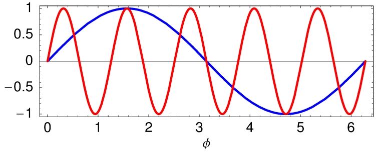
D. Bruß and A. Peres, PRA 61, 030301 (2000).



Relation to other criteria



The Fisher information and entanglement



Some known relations: FI & Entanglement

- Not all entangled states are useful for metrology

  P. Hyllus et al., Phys. Rev. A 82, 012337 (2010)

- But: A high value of the Fisher information signals presence of multiparticle entanglement

  G Toth, PRA 85, 022322 (2012), P. Hyllus et al., PRA 85, 022321 (2012)

- Recent methods allow to bound the Fisher information from few measurements, the techniques are similar to entanglement estimation.

  Geza's and Iagoba's talks!

Fisher information

For the unitary evolution

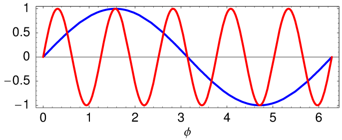
$$U = \exp\{i\varphi H\}$$

the Fisher information of the state $\varrho = \sum_k \lambda_k |k\rangle\langle k|$

$$F(\varrho, A) = 2\sum_{\alpha,\beta} \frac{(\lambda_\alpha - \lambda_\beta)^2}{\lambda_\alpha + \lambda_\beta} |\langle\alpha|A|\beta\rangle|^2$$

bounds the precision

$$\Delta(\varphi) \geq \frac{1}{\sqrt{F}}$$



Observation

Key observation

- The FI is concave in the state
- The FI equals the variance for pure states.

G. Toth, D. Petz, PRA 87, 032324 (2013), S. Yu, arXiv:1302.5311

Question: Can we use this to derive entanglement criteria as we can do with covariance matrices?

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Advantages

- The criterion will directly detect useful entanglement.
- The techniques from the CMC criterion can be used, up to some sign flips.

The FIM

Consider observables M_k and the matrix

$$F(\varrho, M_k) = 2 \sum_{\alpha,\beta} \frac{(\lambda_\alpha - \lambda_\beta)^2}{\lambda_\alpha + \lambda_\beta} \langle\alpha|M_i|\beta\rangle\langle\beta|M_j|\alpha\rangle$$

The diagonal entries are the FI from the M_k

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The diagonal entries are the FI from the M_k

Convexity property

From the convexity of the FI it follows that if $\varrho = \sum_k p_k \varrho_k$, then

$$F(\varrho) \leq \sum_k p_k F(\varrho_k).$$

Interpretation. The FI decreases under mixing the states.

Properties of the FIM

- The FIM is a $d^2 \times d^2$ matrix with nonnegative eigenvalues.
- For a pure state the rank is $r(F) = 2(d-1)$, the rank can maximally be $r(F) = d^2 - d$.
- The eigenvalues of the FIM are

$$\eta_k = 2\frac{(\lambda_\alpha - \lambda_\beta)^2}{\lambda_\alpha + \lambda_\beta}$$

- Unless $d = 2$, the FIM determines ϱ completely.

First criterion

We take the $2d^2$ observables $\{M_k\} = \{A_k \otimes \mathbb{1},\, \mathbb{1} \otimes B_k\}$. Then

$$F = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix},$$

where $A = F(\varrho_A, \{A_k\})$ and $B = F(\varrho_B, \{B_k\})$.

FMC

If ϱ is separable, then there exists

$$f_A = \sum_k p_k F(|a_k\rangle\langle a_k|) \text{ and } f_B = \sum_k p_k F(|b_k\rangle\langle b_k|)$$

such that

$$F(\varrho) \leq \begin{bmatrix} f_A & 0 \\ 0 & f_B \end{bmatrix} =: f_A \oplus f_B.$$

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Problem

For two-qubit states of the form

$$\varrho = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{1}}{4}$$

the f_a, f_b exist already for $p = 0.66$
$\Rightarrow$ The criterion cannot be very strong, regardless of the evaluation.

Second criterion

- We take the d^4 observables $\{M_{ij}\} = \{A_i \otimes \mathbb{1} + \mathbb{1} \otimes B_j\}$. Again this matrix has a block structure.

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- For two-qubit states of the form

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  the matrices exist only for $p < 0.33$
  $\Rightarrow$ Looks better, but so far we have no direct evaluation criterion.

Conclusion

- Covariance matrices are powerful tools to characterize entanglement
- The CMC unifies several other known criteria
- The Fisher information matrix may also be a useful tool, but we have to work more.

Is there a relation to R. Augusiak, arXiv:1506.08837 ?

- O. Gühne, P. Hyllus, O. Gittsovich, J. Eisert
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Thanks to ...

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