Information geometry and metrology in open quantum dynamical systems

Mădălin Guţă

School of Mathematical Sciences
University of Nottingham

M.G., J. Kiukas, arXiv:1601.04355

M.G., J. Kiukas, Commun. Math. Phys. (2015) arXiv:1402.3535

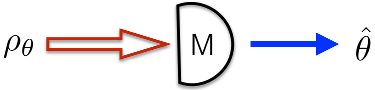
K. Macieszczak, M.G. I. Lesanovsky, J. P. Garrahan, Phys. Rev. A (2016) arXiv:1411.3914

Recent Advances in Quantum Metrology
Warsaw 2016

Outline

- Quantum parameter estimation
  - Standard vs Heisenberg scaling
  - Discrete time Markov dynamics
- Multiparameter estimation of input-output systems in continuous time
  - Identifiable parameters for ergodic dynamics
  - Quantum Fisher information as Markov covariance of generators
  - Gaussian approximation of output state at large times
- Enhanced metrology for output phase
  - at dynamical phase transition
  - near dynamical phase transition

Quantum parameter estimation



- Encode unknown parameter θ into quantum state ρ_θ
- Goal: estimate θ by measuring system in state ρ_θ
- Quantum Cramér-Rao bound for the mean square error

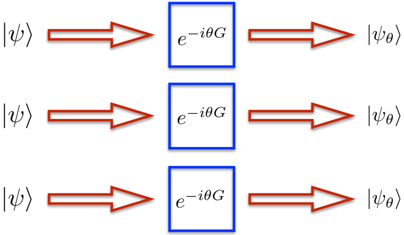
$$\mathbb{E}\left[(\hat{\theta}-\theta)^2\right] \geq \frac{1}{F_\theta}$$

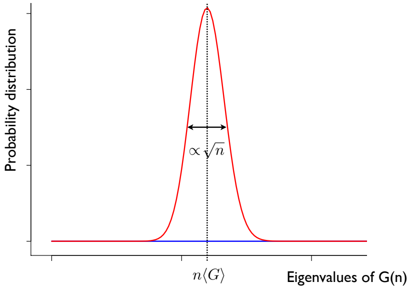
- Quantum Fisher information[1] for unitary rotation families $|\psi_\theta\rangle := e^{-i\theta G}|\psi\rangle$

$$\begin{aligned} F_\theta &= 4\left[\|\dot{\psi}_\theta\|^2 - |\langle\dot{\psi}_\theta|\psi_\theta\rangle|^2\right] \\ &= 4\mathrm{Var}_\psi(G) = 4\left[\left\langle\psi\,\middle|\,G^2\,\middle|\,\psi\right\rangle - \langle\psi|G|\psi\rangle^2\right] \end{aligned}$$

[1] A. Holevo. Probabilistic and Statistical Aspects of Quantum Theory 1982; S. L. Braunstein and C. M. Caves, P.R.L. 1994

Standard scaling as Central Limit behaviour

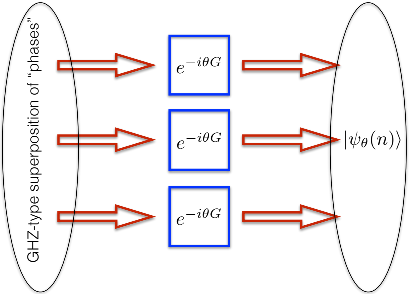


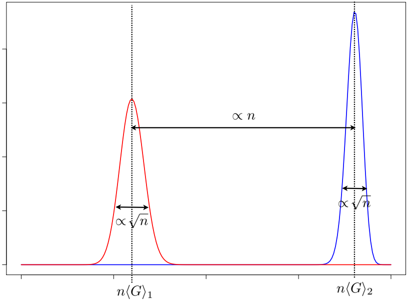


- total generator $G(n) := G^{(1)} + \cdots + G^{(n)}$
- Standard scaling: quantum Fisher information scales linearly in n

$$F_\theta(n) = 4\mathrm{Var}(G(n)) = 4n\mathrm{Var}(G)$$

Heisenberg scaling and phase transitions





- Output: GHZ-type superposition of "Gaussian phases"

$$|\psi_\theta(n)\rangle = |0\rangle^{\otimes n} + e^{-in\theta}|1\rangle^{\otimes n}$$

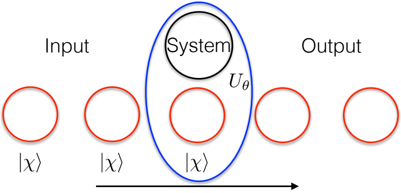
- Heisenberg scaling: quantum Fisher information scales quadratically in n

$$F_\theta(n) = 4\mathrm{Var}(G(n)) \propto n^2$$

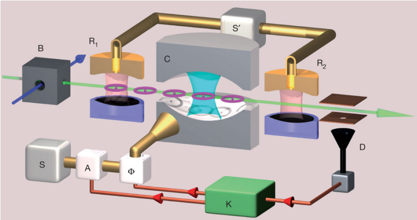
- Bimodal distribution of $G(n)$ reminiscent of phase transitions[2]

[2] P. Zanardi et al., Phys. Rev. A 78, 042105 (2008)

Discrete Markov dynamics = channels with memory



Successive interactions with (memory) system via unitary U_θ



Feedback control of cavity state in the atom maser
[C. Sayrin *et al*, Nature 2011]

- Setup generalises both I.I.D. and GHZ examples depending on choice of U_θ
- Matrix product system-output state

$$|\Psi_\theta^{s+o}(n)\rangle = \sum_{i_1,\ldots,i_n} K_\theta^{i_n} \ldots K_\theta^{i_1} |\chi\rangle \otimes |i_n\rangle \otimes \cdots \otimes |i_1\rangle, \qquad K_\theta^i = \langle i | U_\theta | \psi \rangle$$

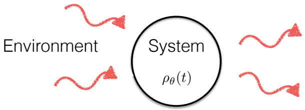
- System identification problem[3]: estimate θ by measuring the output state

[3] M.G., J. Kiukas, Commun. Math. Phys. 2015, M. Cramer *et al*, Nat. Commun. 2010

Outline

- Quantum parameter estimation
  - Standard vs Heisenberg scaling
  - Discrete time Markov dynamics
- Multiparameter estimation of input-output systems in continuous time
  - Identifiable parameters for ergodic dynamics
  - Quantum Fisher information as Markov covariance of generators
  - Gaussian approximation of output state at large times
- Enhanced metrology for output phase
  - at dynamical phase transition
  - near dynamical phase transition

Markovian quantum open systems



- Dissipative evolution of open system

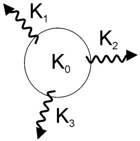
$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = -i[H, \rho(t)] + \sum_i L^i \rho(t) L^{i*} - \frac{1}{2}\{\rho(t), L^{i*}L^i\}$$

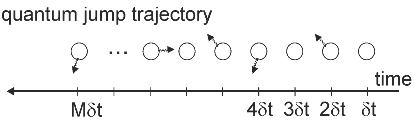
- Ergodicity: system converges to stationary state ρ_{ss} $(\mathcal{L}\rho_{ss} = 0)$

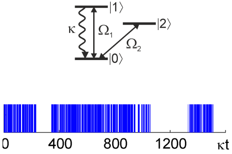
$$\rho(t) = e^{t\mathcal{L}}\rho_{in} \longrightarrow \rho_{ss}$$

- Estimate unknown "dynamical parameters" $\theta \mapsto D_\theta = (H_\theta, L^i_\theta)$ by measuring environment
  - system may not be accessible (e.g. in quantum control applications)
  - system would need to be initialised repeatedly
  - Information about dynamical parameters "leaks" continuously into the environment

Quantum trajectories = unravelling the master dynamics





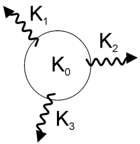


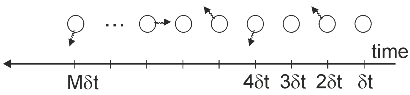
- Monitoring the environment produces jump trajectories with infinitesimal Kraus operators
  - "no emission": $K_0 = e^{-i\delta t H_\theta}\sqrt{1-\delta t \sum_j L_\theta^{j*} L_\theta^j}$
  - "emission" in channel j: $K_j = e^{-i\delta t H_\theta}\sqrt{\delta t} L_\theta^j$

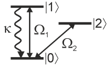
[4] M. Fannes, B. Nachtergale and R. Werner, 1992; D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, 2007

[5] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Springer Birkhäuser, 1992

Quantum trajectories = unravelling the master dynamics







- Monitoring the environment produces jump trajectories with infinitesimal Kraus operators
  - "no emission": $K_0 = e^{-i\delta t H_\theta} \sqrt{1 - \delta t \sum_j L_\theta^{j*} L_\theta^j}$
  - "emission" in channel j: $K_j = e^{-i\delta t H_\theta} \sqrt{\delta t} L_\theta^j$

- System-output state: coherent superposition of quantum trajectories, (continuous) MPS[4]

$$|\psi_\theta^{s+o}(t)\rangle = U_\theta(t)|\psi_{in}^{s+o}\rangle = \sum_{j_1,\dots,j_n} K_{j_n} \dots K_{j_1}|\chi\rangle \otimes |j_n \dots j_1\rangle, \qquad n = t/\delta t$$

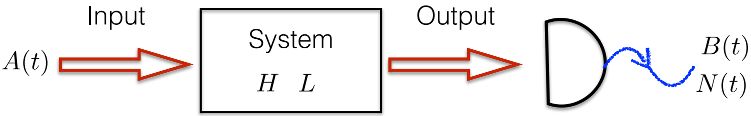
- Unitary dynamics: singular coupling with incoming input fields (Q Stoch Diff Eq[5])

$$dU_\theta(t) = \left(-iH_\theta dt + \sum_i L_\theta^i dA_i^*(t) - L_\theta^{i*} dA_i(t) - \frac{1}{2} L_\theta^{i*} L_\theta^i dt \right) U_\theta(t)$$

[4] M. Fannes, B. Nachtergale and R. Werner, 1992; D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, 2007

[5] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Springer Birkhäuser, 1992

System identification and metrology with input-output open systems



- System identification problem: if $\theta \to (H_\theta, L_\theta)$, estimate θ by measuring the output[6]
  - which parameters can be identified ?
  - How does the output QFI scale with time t ?
  - How does this related to dynamical properties, e.g. ergodicity, spectral gap...?

- Metrological power of output
  - is the Heisenberg limit achievable?
  - what is the short and long time behaviour ?
  - can the output be used as general purpose metrological probe state ?

[6] H. Mabuchi Quant. Semiclass. Optics (1996); J. Gambetta and H. M. Wiseman Phys. Rev. A (2001); S. Gammelmark and K. Molmer Phys. Rev. A (2013)...

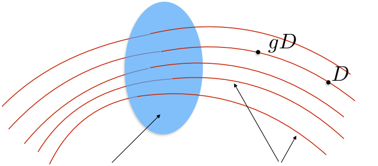
Equivalence classes of dynamical parameters with identical output states

- Stationary output state for dynamical parameter $D = (H, L)$

$$\rho_D^{out}(t) = \mathrm{Tr}_s \left[U_D(t) \; \rho_{ss}^D \otimes |\Omega\rangle\langle\Omega| \; U_D^*(t) \right]$$

- Output state invariant under action of group $G = PU(d) \times \mathbb{R}$ on space of parameters $\mathcal{D}$

$$D = (H, L) \longmapsto gD = (W^* H W + c\mathbf{1}, W^* L W), \quad g = (W, c)$$



Theorem (Equivalence classes & identifiable parameters)

Let $D = (H, L)$ and $D' = (H', L')$ be dynamic parameters of ergodic dynamics. Then

1) $\rho_D^{out}(t) = \rho_{D'}^{out}(t)$ for all $t > 0$, if and only if $D' = gD$ for some $g \in G$.

2) Identifiable parameters form quotient manifold $\mathcal{P} := \mathcal{D}/G$ with $G = PU(d) \times \mathbb{R}$

Idea of the proof

- Overlap of the two system-output states

$$\left\langle \Psi_D^{s+o}(t) \mid \Psi_{D'}^{s+o}(t) \right\rangle = \mathrm{Tr}\left[e^{t\mathcal{L}_{D,D'}}(\rho_{in}) \right]$$

- "Off-diagonal master generator"

$$\mathcal{L}_{D,D'}(\rho) = -i(H\rho - \rho H') + L\rho L'^{*} - \frac{1}{2}\left(L^{*}L\rho + \rho L^{*\prime} L' \right)$$

- Two alternatives:

A) $D' = gD \Leftrightarrow \mathcal{L}_{D,D'}$ has an imaginary eigenvalue $\Leftrightarrow$ identical outputs

B) $D' \neq gD \Leftrightarrow \mathcal{L}_{D,D'}$ is "stable" $\Leftrightarrow$ "orthogonal" outputs

Generator of parameter change in system+output state

- Model dynamics with unknown parameter $\theta \in \mathbb{R}^m$

$$D_\theta = (H_\theta, L_\theta) \quad \longrightarrow \quad \left|\Psi_\theta^{\rm s+o}(t)\right\rangle = U_\theta(t)|\varphi \otimes \Omega\rangle$$

- Tangent vector at D_θ corresponding to changes in component θ_a

$$\dot{D}_{\theta,a} = (\dot{H}_{\theta,a}, \dot{L}_{\theta,a}) = \left(\frac{\partial H}{\partial \theta_a}, \frac{\partial L}{\partial \theta_a}\right)$$

- Generator of parameter change for component θ_a

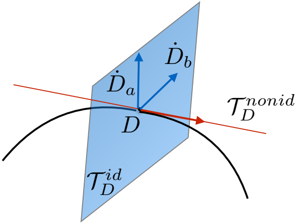
$$\frac{\partial}{\partial \theta_a}\left|\Psi_\theta^{\rm s+o}(t)\right\rangle = \dot{U}_{\theta,a}(t)|\varphi \otimes \Omega\rangle = U_\theta(t) G_{\theta,a}(t)|\varphi \otimes \Omega\rangle$$

- Generator is a quantum stochastic integral (fluctuation operator)

$$G_{\theta,a}(t) \quad := \quad \sqrt{t}\mathbb{F}_t(\dot{D}_{\theta,a}) = \int_0^t \dot{L}_{\theta,a}(s) dA^*(s) - i\mathcal{E}_D(\dot{D}_{\theta,a})(s) ds$$

$$\mathcal{E}_D(\dot{D}) \quad := \quad \dot{H} + \mathrm{Im}(\dot{L}^* L) - \mathrm{Tr}\left[\rho_{ss}^D(\dot{H} + \mathrm{Im}(\dot{L}^* L))\right]\mathbf{1}$$

Quantum Fisher information as covariance of generators



Theorem

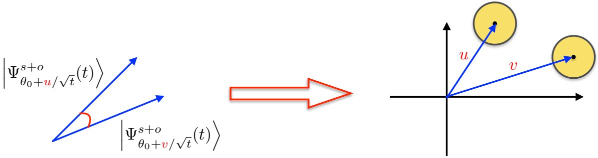
The quantum Fisher information matrix $F_{a,b}(t) = 4\mathrm{Re}\left\langle G^*_{\theta,a}(t)\cdot G_{\theta,b}(t)\right\rangle$ *grows linearly in* t *with rate* $F_{a,b}$ *given by the asymptotic Markov covariance of fluctuators*

$$\begin{aligned} F_{a,b} &= 4\mathrm{Re}\left(\dot{D}_{\theta,a},\dot{D}_{\theta,b}\right)_D \\ &:= 4\mathrm{Re}\,\mathrm{Tr}\left[\rho_{ss}\left(\dot{L}_{\theta,a} - i[L_\theta, \mathcal{L}^{-1}\circ\mathcal{E}_{\mathsf{D}}(\dot{\mathsf{D}}_{\theta,a})]\right)^*\cdot\left(\dot{L}_{\theta,b} - i[L_\theta,\mathcal{L}^{-1}\circ\mathcal{E}_{\mathsf{D}}(\dot{\mathsf{D}}_{\theta,b})]\right)\right]. \end{aligned}$$

The tangent space decomposes into identifiable and unidentifiable subspaces $\mathcal{T}_D = \mathcal{T}_D^{id}\oplus\mathcal{T}_D^{nonid}$

- $\mathcal{T}_D^{nonid} := \{\dot{D} : \dot{D} = i[K,D] + c(\mathbf{1},0)\} \quad\longrightarrow\quad (\dot{D},\dot{D}')_D = 0$
- $\mathcal{T}_D^{id} = \{\dot{D} : \mathcal{E}_D(\dot{D}) = 0\} \quad\longrightarrow\quad (\dot{D},\dot{D}')_D = \mathrm{Tr}(\rho^D_{ss}\dot{L}^*\dot{L}')$
- $F_{a,b}$ *defines a Riemannian metric on* $\mathcal{P} = \mathcal{D}/G$

Gaussian approximation for (system +) output state[7]



- Parameter uncertainty $\approx t^{-1/2} \Rightarrow$ interesting statistical features are local: $\theta = \theta_0 + u/\sqrt{t}$

$$D_{\theta_0+u/\sqrt{t}} = D_{\theta_0} + \frac{1}{\sqrt{t}}\dot{D}_u + O(t^{-1}) = D_{\theta_0} + \frac{1}{\sqrt{t}}\sum_a u_a \dot{D}_{\theta_0,a} + O(t^{-1})$$

- CCR algebra over $\mathcal{T}^{id}_D$ (continuous variable system) with commutation relations

$$W(u)W(v) = e^{-i\mathrm{Im}(\dot{D}_u,\dot{D}_v)_D} W(u+v)$$

- Convergence to coherent states $|u\rangle := W(u)|0\rangle$

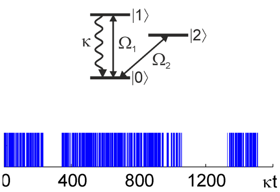
$$\lim_{t\to\infty}\left\langle \Psi^{s+o}_{\theta_0+u/\sqrt{t}}(t) \,\middle|\, \Psi^{s+o}_{\theta_0+v/\sqrt{t}}(t)\right\rangle = e^{-\frac{1}{2}\|\dot{D}_u-\dot{D}_v\|^2_D} = \langle u|v\rangle$$

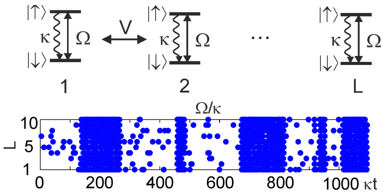
[7] M.G., J. Kiukas, arXiv:1601.04355

Outline

- Quantum parameter estimation
  - Standard vs Heisenberg scaling
  - Discrete time Markov dynamics
- Multiparameter estimation of input-output systems in continuous time
  - Identifiable parameters for ergodic dynamics
  - Quantum Fisher information as Markov covariance of generators
  - Gaussian approximation of output state at large times
- Enhanced metrology for output phase
  - at dynamical phase transition
  - near dynamical phase transition

Counting statistics and dynamical phase transitions[8]

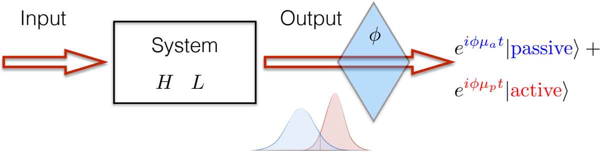




- If $\mathcal{L}$ is ergodic (spectral gap $\Delta\lambda := -\mathrm{Re}\lambda_2 > 0$) then
  - system converges to stationary state $\rho(t) = e^{t\mathcal{L}}(\rho_{\mathrm{in}}) \xrightarrow{t\to\infty} \rho_{ss}$
  - Counting operator $N(t)$ has normal fluctuations ($\Delta N(t) \propto \sqrt{t}$) around mean $t\mu$
- If $\mathcal{L}$ is near phase transition ($\Delta\lambda \approx 0$) then
  - slow convergence to stationarity, long correlation time $\tau = 1/\Delta\lambda$
  - intermittent trajectories, counting operator $N(t)$ has bimodal distribution up to times τ
- If $\mathcal{L}$ has degenerate stationary states then
  - infinite correlation times
  - counting operator $N(t)$ remains bimodal all times and variance increases as t^2

[8] J. Garrahan, I. Lesanovsky, P.R.L. (2010); I. Lesanovsky, M. van Horssen, M. G., J. P. Garrahan, P.R.L. (2013)

Phase estimation: Heisenberg limit at the DPT



- First order phase transition: system with two "stationary phases" ($\mathcal{H} = \mathcal{H}_i \oplus \mathcal{H}_a$) with different emission rates $\mu_i \neq \mu_a$
- Initial state: superposition $\sqrt{p_i}|\chi_i\rangle + \sqrt{p_a}|\chi_a\rangle$ with $|\chi_{a,i}\rangle \in \mathcal{H}_{i,a}$
- GHZ-type system-output state with generator $N(t)$

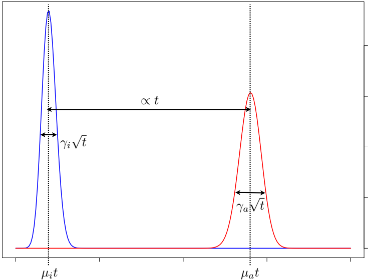
  $$|\psi_\phi(t)\rangle = e^{i\phi N(t)}|\psi(t)\rangle \approx \sqrt{p_i}e^{i\phi\mu_i t}|\psi_i(t)\rangle + \sqrt{p_a}e^{i\phi\mu_a t}|\psi_a(t)\rangle$$

- Heisenberg limit:

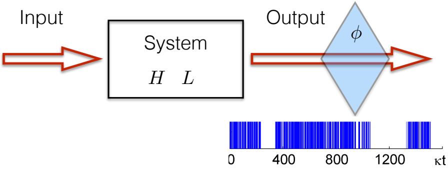
  $$F(t) = 4\text{Var}(N(t)) \approx t^2 p_i p_a (\mu_a - \mu_i)^2$$

- must measure sys+out to achieve QFI

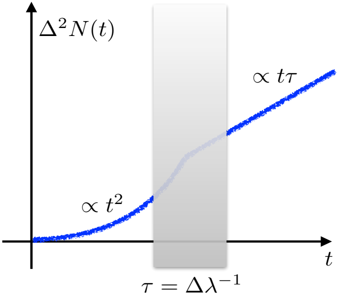
  [K. Macieszczak, M.G. I. Lesanovsky, J. P. Garrahan Phys. Rev. A 2016]



Phase estimation: QFI time behaviour near phase transition



- System near first order DPT: metastability $\Longrightarrow$ counting trajectories exhibit intermittency
- Short time $(t \ll \tau)$ distribution of generator $N(t)$ is bimodal $\Longrightarrow$ quadratic growth of QFI
- Long time $(t \gg \tau)$ ergodicity and normal fluctuations win $\Longrightarrow$ linear growth of QFI



Conclusions and Outlook

- System identification: ergodic systems can be completely identified from output, up to unitary "change of coordinates"
- Asymptotic normality: output state is asymptotically Gaussian with quantum Fisher information equal to the "Markov variance of the generator"
- Systems at DPT exhibit Heisenberg scaling of estimation precisions
- Near phase transition: initial quadratic increase, then linear (possibly) with large constant

- Ongoing / future work
  - System identification and metrology for linear systems
  - Metastability theory for quantum open systems
  - General quantum Markov CLT
  - use of coherent feedback in system identification
  - design of better input states

References

M. Guta, J. Kiukas
Information geometry and local asymptotic normality for multi-parameter estimation of quantum Markov dynamics
ArXiv:1601.04355

K. Macieszczak, M. Guta, I. Lesanovsky, J. P. Garrahan
Dynamical phase transitions as a resource for quantum enhanced metrology
Phys. Rev. A 93, 022103 (2016)

C. Catana, L. Bouten and M. Guta
Fisher informations and local asymptotic normality for continuous-time quantum Markov processes
J. Phys. A, 48, 365301 (2015)

M. Guta, N. Yamamoto
System Identification for Passive Linear Quantum Systems
IEEE Trans. Automatic Control, (2016)

M. van Horssen and M. Guta:
Sanov and Central Limit Theorems for output statistics of quantum Markov chains
J. Math. Phys. 56, 022109 (2015)

M. Guta, J. Kiukas
Equivalence classes and local asymptotic normality in system identification for quantum Markov chains
Commun. Math. Phys. (2014)

I. Lesanovsky, M. van Horssen, M. Guta, J. P. Garrahan
Characterization of dynamical phase transitions in quantum jump trajectories beyond the properties of the stationary state
Phys. Rev. Lett. 110, 150401 (2013)

C. Catana, M. van Horssen, M. Guta:
Asymptotic inference in system identification for the atom maser
Phil. Trans. R. Soc. A 370, 5308-5323 (2012)