Relationship between communication and quantum metrology

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Communication Theory

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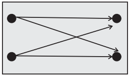

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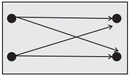

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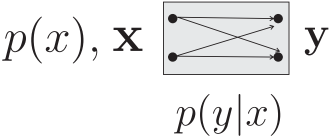

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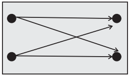
$p(x)$, $\mathbf{x}$  $\mathbf{y}$

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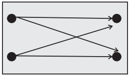
$p(x)$, $\mathbf{x}$  $\mathbf{y}$

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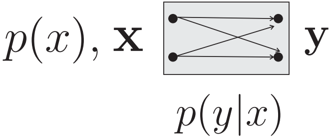

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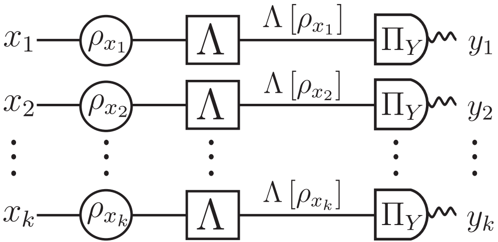

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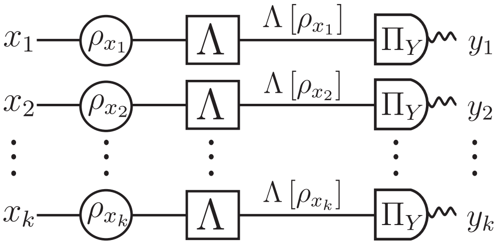

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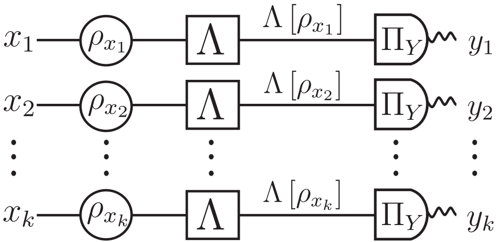

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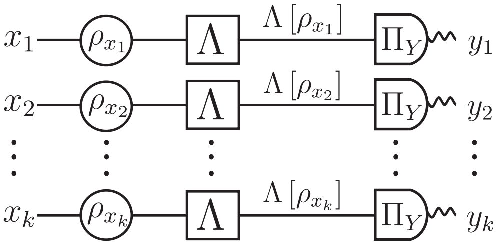

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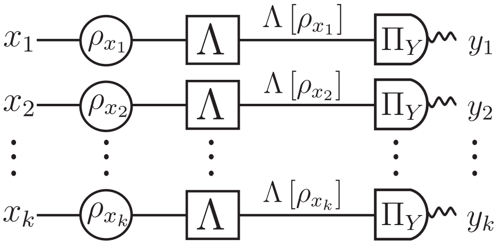

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$$C^{(1,1)} \leq \chi = S\left(\int dxp(x)\rho_x\right) - \int dxp(x)S(\rho_x)$$

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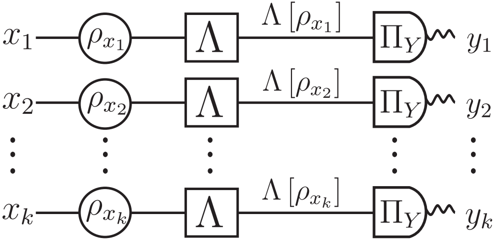
- General scheme



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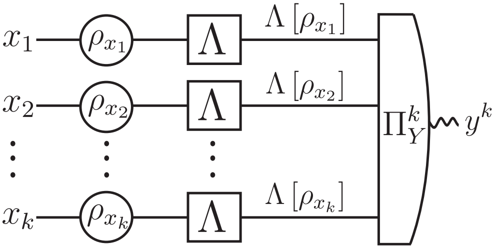

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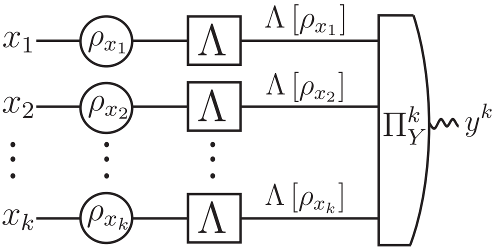

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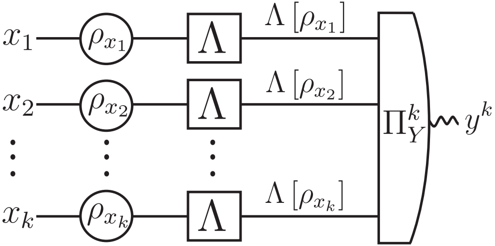

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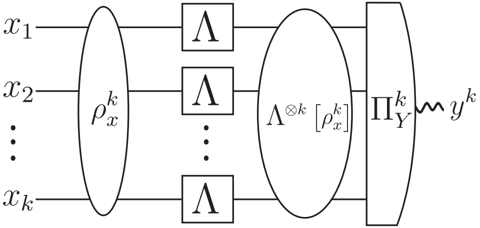

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- Input superadditivity: use entangled states $\rightarrow$ $C^{(k,\infty)} \geq C^{(1,\infty)}$ (Hastings, Nat. Phys. (2009)).

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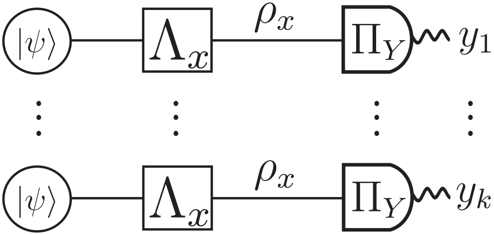

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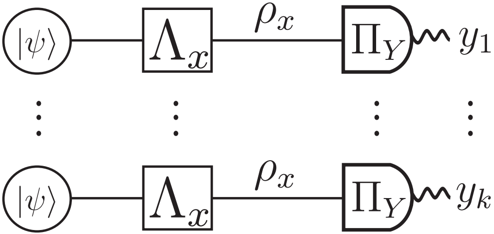

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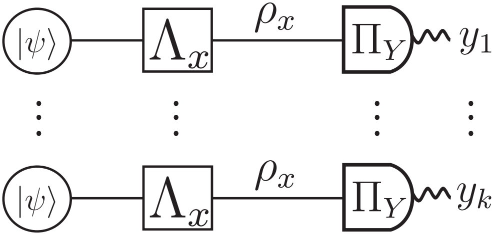

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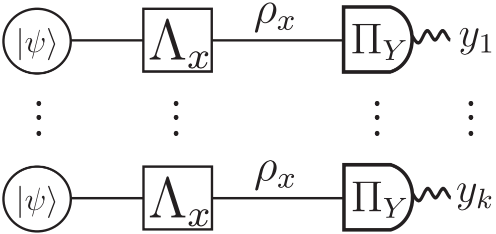

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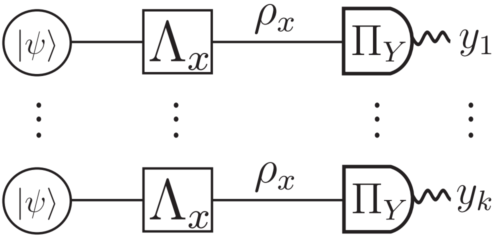

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- Bound on precision optimized over all unbiased estimators.

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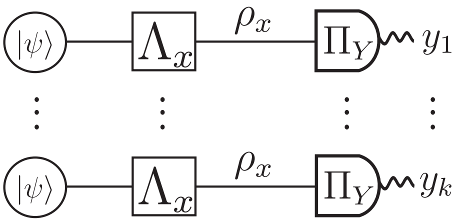

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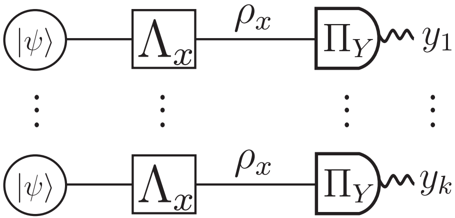

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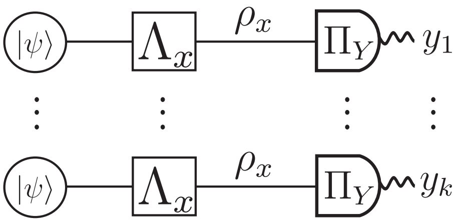

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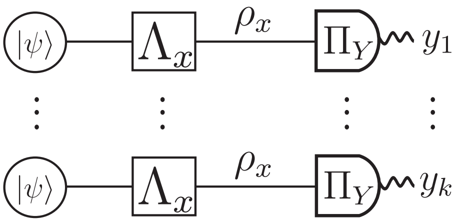

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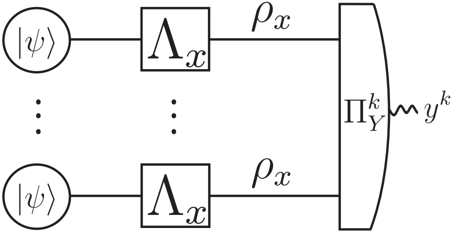

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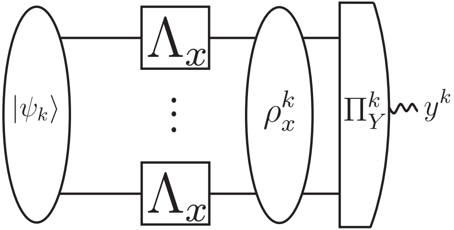

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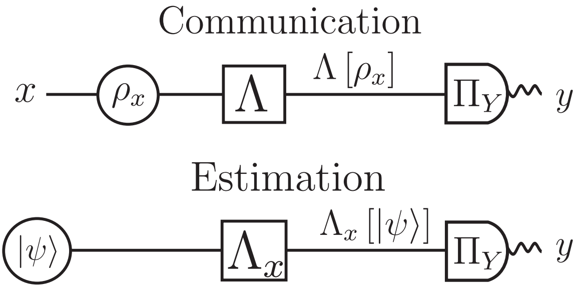

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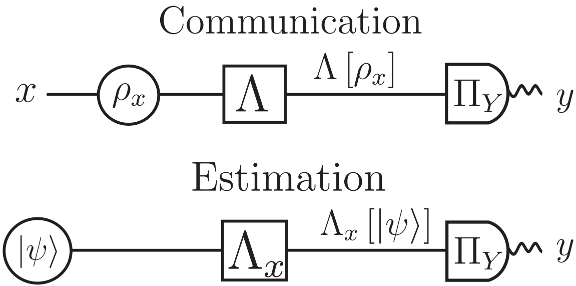

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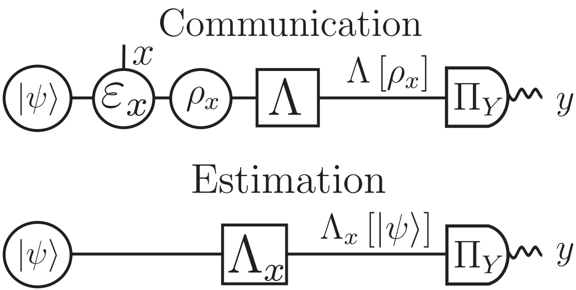

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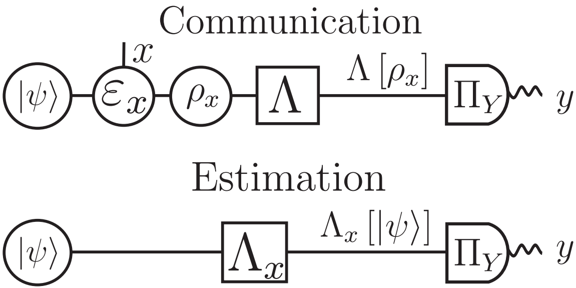

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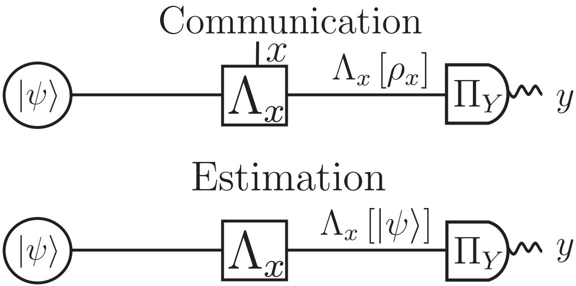

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- Communication $\approx$ estimation.

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- **Warning**: Do not increase k to much! $\sigma^2 \ll 1/F(\bar{x})$

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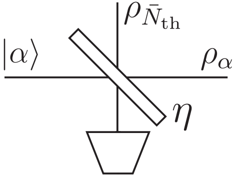

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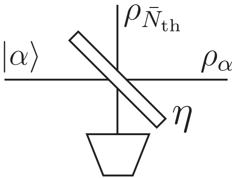

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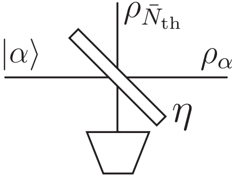

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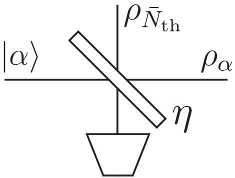

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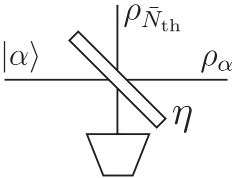

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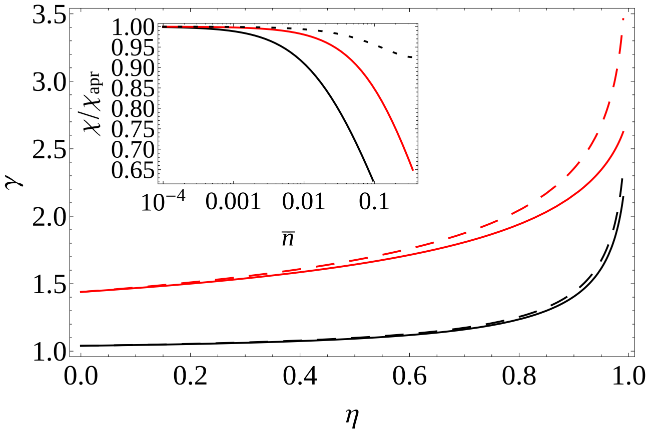

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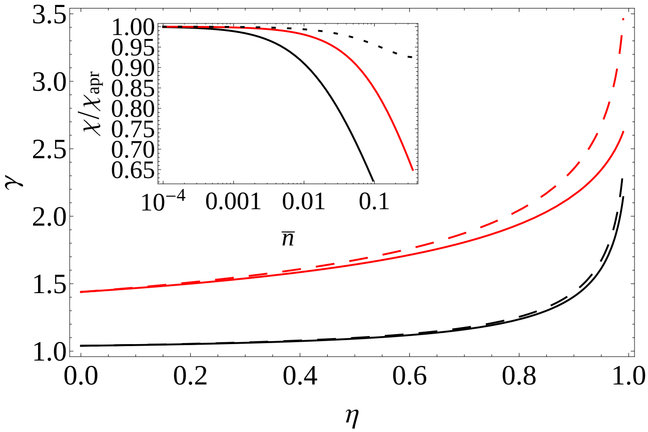

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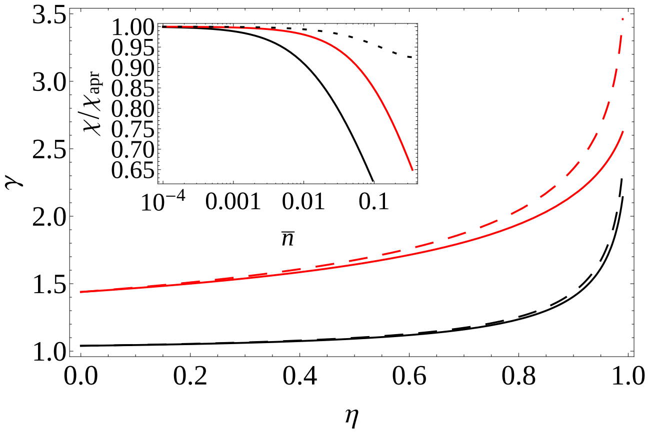

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- Inset - convergence of the approximation: $\bar{N}_{\mathrm{th}} = 0,\ \eta = 0.9$ - dotted; $\bar{N}_{\mathrm{th}} = 0.1,\ \eta = 0.5$ - red; $\bar{N}_{\mathrm{th}} = 1,\ \eta = 0.99$ - black, solid.

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