RANDOM SYMMETRIC STATES FOR ROBUST QUANTUM METROLOGY

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ICFO – The Institute of Photonic Sciences, Barcelona, Spain

“Recent Advances in Quantum Metrology”, University of Warsaw, March 2016

OUTLINE OF THE TALK

1. Connection between *geometry of quantum states* and their *metrological properties*

[**Continuity of Quantum Fisher Information** (QFI)]

i. Natural link between the ***QFI*** and the ***Geometric Measure of Entanglement*** (E_{G}).

ii. State (sequences) with ***vanishing entanglement properties*** with system size can yield precision scaling ***arbitrary close to the Heisenberg Limit***.

arxiv:1506.08837

2. Quantum metrology with random (typical) states

[iso-spectral quantum states sampled uniformly from the Haar measure on a unitary group]

i. Random states of N **distinguishable particles** (*qudits*) are ***useless*** for quantum metrology (despite possessing on average high entanglement, $E_{\mathrm{G}} \approx 1$, and even allowing for **LU optimisation**).

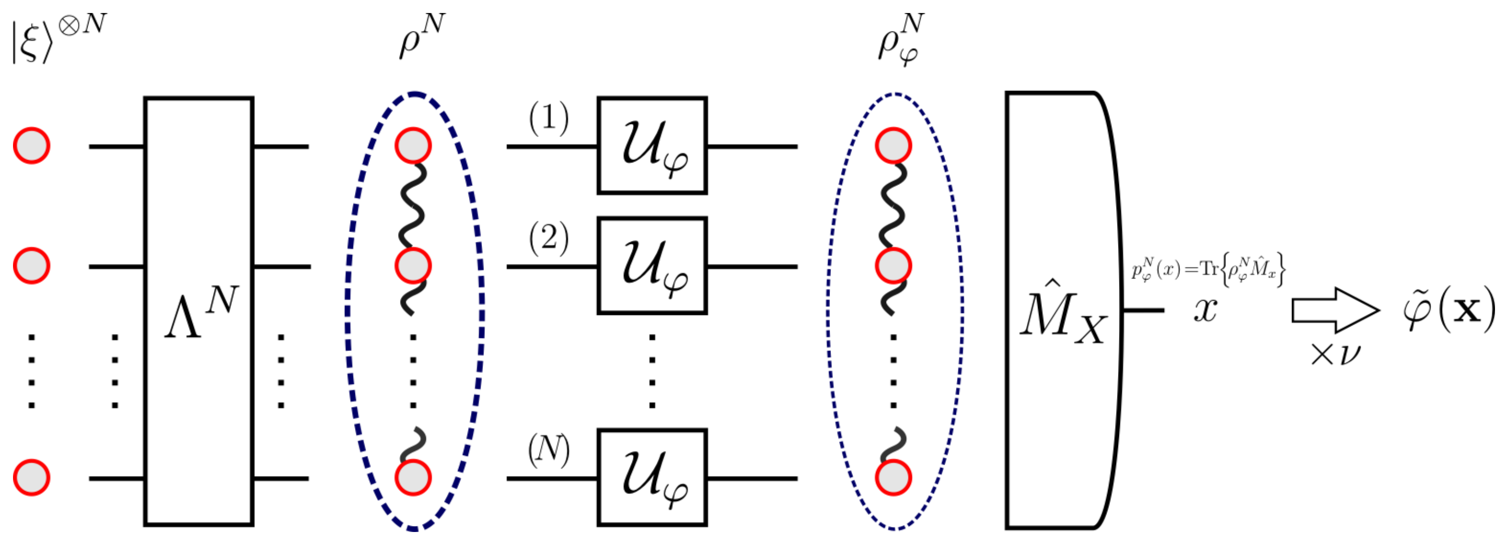
ii. Random states of N **symmetric particles** (*d-mode bosons*) typically ***achieve the Heisenberg Limit***.

- They are **robust against mixing noise** that ("non-exponentially") increases with system size.
- They are **robust against particle losses** that ("sub-linearly") increase with system size.

iii. Random states of N **pure symmetric particles** (bosons) typically ***achieve the Heisenberg Limit*** with **measurement fixed** to the (*Mach-Zehnder*) **interferometric** one with photon counting.

- They can be **simulated efficiently** with short **random optical circuits** generated from a set of **three types of beamsplitters** and a **single non-linear** (Kerr-type) **transformation**.

QUANTUM METROLOGY PROTOCOL



Unitary encoding of the parameter:

$$\mathcal{U}_\varphi[\varrho] = U_\varphi \varrho U_\varphi^\dagger \qquad \text{with} \quad U_\varphi = \mathrm{e}^{-\mathrm{i}\hat{h}\varphi} \qquad \left\|\hat{h}\right\| \leq \frac{1}{2}, \text{ e.g., for qubits } \hat{h} := \frac{1}{2}\hat{\sigma}_z$$

[e.g. (*squeezed*) photons in Mach-Zehnder interferometry, (*spin-squeezed*) atoms in Ramsey spectroscopy]c

Ultimate bound on precision of estimation in the limit of sufficiently large statistics ($\nu \to \infty$):

Quantum Cramer-Rao Bound

$$\nu\, \Delta^2 \tilde{\varphi} \underset{(\nu \to \infty)}{\geq} \frac{1}{F_Q[\rho^N]}$$

Quantum Fisher Information (QFI)

$$F_Q[\rho^N] := 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle \psi_k | \hat{H} | \psi_l \rangle|^2 \qquad \rho^N = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \qquad \hat{H} = \sum_{n=1}^{N} \hat{h}^{(n)}$$

- ***Local (frequentist) estimation*** with sufficiently large statistics (in contrast to the *Bayesian one-shot approach*).
- ***Optimised over all measurements***/inference strategies (for **fixed measurement** need to consider **classical FI**).
- ***Parameter-independence of QFI*** due to unitary encoding (***not* true for fixed measurement and classical FI**).
- ***Fix encoding Hamiltonian*** and study properties of states, but in the *Heisenberg picture* analysis of Hamiltonians.

Asymptotic Role of Entanglement in Quantum Metrology

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Antonio Acin, Remigiusz Augusiak, Manabendra Nath Bera, Janek Kolodynski, Maciej Lewenstein, Alexander Streltsov

Continuity of QFI on quantum states:

$$\underset{\rho^N,\sigma^N\in\mathcal{B}(\mathcal{H}^{\otimes N})}{\forall}: \quad \boxed{\left|F_{\mathrm{Q}}\left[\rho^N\right]-F_{\mathrm{Q}}\left[\sigma^N\right]\right| \leq \xi\sqrt{1-\mathcal{F}(\rho^N,\sigma^N)^2}\,N^2} \quad \leq \xi\;\mathcal{D}_{\mathrm{B}}(\rho^N,\sigma^N)\,N^2,$$

where $\mathcal{F}(\varrho,\sigma):=\mathrm{Tr}\sqrt{\sqrt{\sigma}\varrho\sqrt{\sigma}}$ is the Uhlmann fidelity and $\mathcal{D}_{\mathrm{B}}(\varrho,\sigma):=\sqrt{2(1-\mathcal{F}(\varrho,\sigma))}$ is the Bures distance.

Aside for specialists:

In general, we prove $\xi=8$ using ***purification-based definition*** of QFI:

[A. Fujiwara, PRA 63, 042304 (2001); B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nature Phys. 7, 406 (2011)]

If one of the states is *pure* we may tighten the bound to $\xi=6$ via the ***convex-roof-based definition*** of QFI:

[G. Toth and D. Petz, PRA 87, 032324 (2013); S. Yu arXiv:1302.5311 (2013)]

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where $\mathcal{F}(\varrho,\sigma):=\mathrm{Tr}\sqrt{\sqrt{\sigma}\varrho\sqrt{\sigma}}$ is the Uhlmann fidelity and $\mathcal{D}_{\mathrm{B}}(\varrho,\sigma):=\sqrt{2(1-\mathcal{F}(\varrho,\sigma))}$ is the Bures distance.

Being **close** to metrologically ***useful*** states is **good**:

$$F_{\mathrm{Q}}[\rho^N]\sim N^2 \;\Longrightarrow\; F_{\mathrm{Q}}[\sigma^N] \gtrsim (1-\xi\,\mathcal{D}_{\mathrm{B}}(\rho^N,\sigma^N))\,N^2 \;\Longrightarrow\; \mathcal{D}_{\mathrm{B}}(\rho^N,\sigma^N)<\frac{1}{\xi} \;\Longrightarrow\; F_{\mathrm{Q}}[\sigma^N]\sim N^2$$

$$\text{e.g., } \rho^N=\psi^N_{\mathrm{GHZ}} \text{ and } \mathcal{F}>\frac{35}{36} \;\Longrightarrow\; F_{\mathrm{Q}}[\sigma^N]\sim N^2$$

[actually $\mathcal{F}>1/2$ is enough, see methods of *I. Appelaniz et al* (arxiv:1511.05203) developed for Dicke states]

Being **too close** to metrologically ***useless*** states is **bad**: $(\varepsilon>1)$

$$F_{\mathrm{Q}}[\sigma^N]\sim N \;\Longrightarrow\; F_{\mathrm{Q}}[\rho^N] \le N+\xi\,\mathcal{D}_{\mathrm{B}}(\rho^N,\sigma^N)\,N^2 \;\Longrightarrow\; \mathcal{D}_{\mathrm{B}}(\rho^N,\sigma^N)\sim\frac{1}{N^{\varepsilon}} \;\Longrightarrow\; F_{\mathrm{Q}}[\rho^N]\lesssim N^{2-\varepsilon}$$

... but for $(0<\varepsilon<1)$ super-classical scaling despite approaching the useless states.

... **natural** (geometrical) connection to **Geometric Measure of Entanglement**.

GEOMETRIC MEASURE OF ENTANGLEMENT, E_{G}

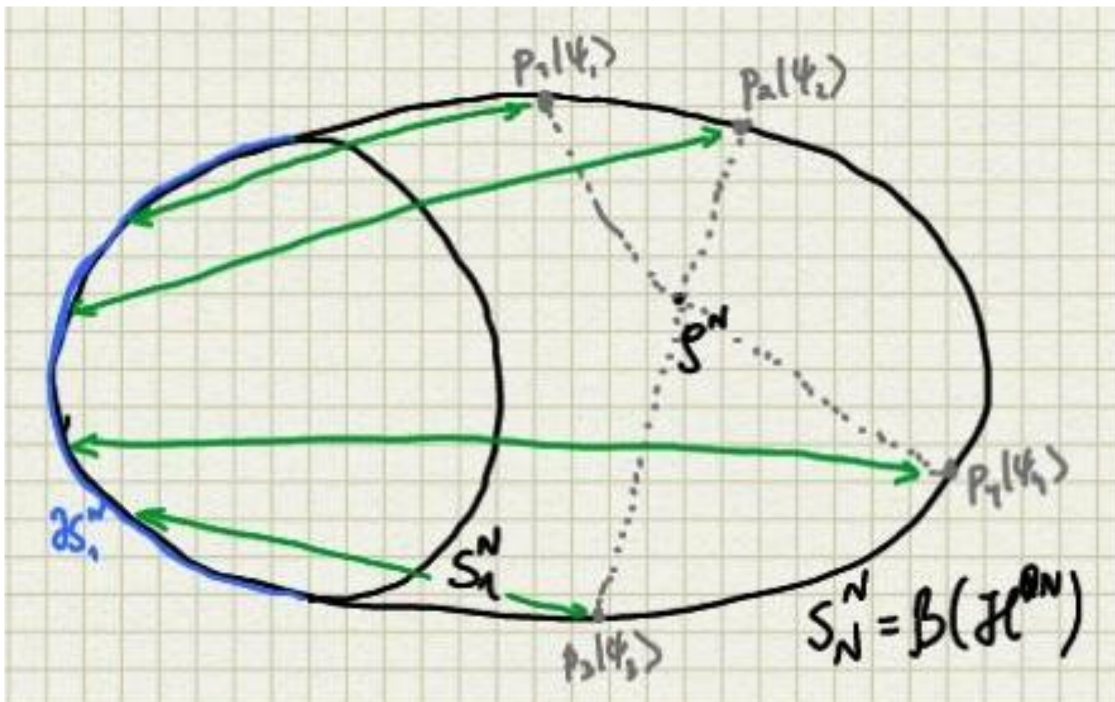
The *geometric measure of entanglement* is defined as:

$$E_{\mathrm{G}}[\rho^N] := \inf_{\{p_i, |\psi_i^N\rangle\}} \sum_i p_i \, E_{\mathrm{G}}[\psi_i^N]$$

with the infimum taken over all ensembles such that $\rho^N = \sum_i p_i |\psi_i^N\rangle\langle\psi_i^N|$, where for any pure state ψ^N :

$$E_{\mathrm{G}}[\psi^N] := 1 - \max_{\phi_{\mathrm{sep}}^N} \left|\langle\phi_{\mathrm{sep}}^N|\psi^N\rangle\right|^2 .$$

Geometric interpretation:



$0 \leq E_{\mathrm{G}}[\rho^N] \leq 1$ $\qquad$ $E_{\mathrm{G}}[\rho_{\mathrm{sep}}^N] = 0$ $\qquad$ $E_{\mathrm{G}}[\psi_{\mathrm{GHZ}}^N] = \frac{1}{2}$

NOTE THE SCALE INDEPENDENCE, i.e., E_G is independent of N for family of same type of states !!!

ARBITRARY CLOSE TO HL WITH VANISHING ENTANGLEMENT

Crucially, this allows us to bound the QFI of a state via its *geometric measure of entanglement* E_{G}:

$$F_{\mathrm{Q}}[\rho^N] \leq \max_{\psi^N_{\mathrm{sep}}} \left\{ F_{\mathrm{Q}}[\psi^N_{\mathrm{sep}}] + \xi \sqrt{1-\mathcal{F}(\rho^N, \psi^N_{\mathrm{sep}})}\, N^2 \right\}$$

$$F_{\mathrm{Q}}[\rho^N] \leq N + \xi \sqrt{E_{\mathrm{G}}[\rho^N]}\, N^2$$

$$\Longrightarrow \quad \boxed{F_{\mathrm{Q}}[\rho^N] \lesssim \sqrt{E_{\mathrm{G}}[\rho^N]}\, N^2}$$

Thus, from the point of view of the asymptotic precision scaling:

$$\nu\, \Delta^2 \tilde{\varphi} \gtrsim \frac{1}{N^{2-\varepsilon}} \quad \Longrightarrow \quad F_{\mathrm{Q}}[\rho^N] \sim N^{2-\varepsilon} \quad \Longrightarrow \quad E_{\mathrm{G}}[\rho^N] \gtrsim \frac{1}{N^{2\varepsilon}}$$

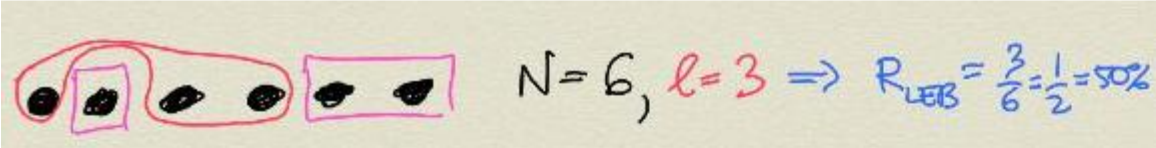
Can be asymptotically vanishing for any ε>0 !!! (only lower bound)

To attain "**exact HL**" E_{G} must be **asymptotically approaching a constant**, but to attain a precision-scaling "**arbitrary close to HL**" E_{G} may be potentially taken to be **arbitrary small for sufficiently large *N***.

On the other hand, the **relative size** of the **Largest Entangled Block**

(the **ratio to the total number of particles**):

$$\rho^N_{(l)} \iff R_{\mathrm{LEB}} = \frac{l}{N}$$



[G. Toth, PRA 85, 022322 (2012); P. Hyllus et al, PRA 85, 022321 (2012)]

$$\boxed{F_{\mathrm{Q}}\left[\rho^N_{(l)}\right] \leq R_{\mathrm{LEB}}\, N^2} \quad \Longrightarrow \quad \nu\, \Delta^2 \tilde{\varphi} \gtrsim \frac{1}{N^{2-\varepsilon}} \quad \Longrightarrow \quad l \gtrsim N^{1-\varepsilon} \quad \Longrightarrow \quad R_{\mathrm{LEB}} \gtrsim \frac{1}{N^{\varepsilon}}$$

Can be asymptotically vanishing for any ε > 0 !!! (only lower bound)

To attain "**exact HL**" R_{LEB} must be **asymptotically approaching a constant**, but to attain a precision-scaling "**arbitrarily close to HL**" R_{LEB} potentially may be taken to be **arbitrary small for sufficiently large *N***.

State that Does the Job

Generalised Werner-type state:

$$\rho_{[l]}^N = p \left|\psi_{\mathrm{GHZ}}^l\right\rangle\left\langle\psi_{\mathrm{GHZ}}^l\right| \otimes \left|0^{N-l}\right\rangle\left\langle 0^{N-l}\right| + (1-p)\frac{\mathbf{1}_N}{2^N} \quad \text{with } 0 \le \mathrm{p} \le 1$$

[L. E. Buchholz, T. Moroder, and O. Gühne, arXiv:1412.7471]

$$E_{\mathrm{G}}\left[\rho_{[l]}^N\right] \le \frac{p}{2}, \quad R_{\mathrm{LEB}}\left[\rho_{[l]}^N\right] \le \frac{l}{N} \qquad F_{\mathrm{Q}}\left[\rho_{[l]}^N\right] \underset{N\to\infty}{\approx} p\,l^2 = 2E_{\mathrm{G}}R_{\mathrm{LEB}}^2 N^2$$

$$p \sim \frac{1}{N^{\varepsilon_1}},\ l \sim N^{1-\varepsilon_2} \quad \Rightarrow \quad \boxed{E_{\mathrm{G}} \sim \frac{1}{N^{\varepsilon_1}}, R_{\mathrm{LEB}} \sim \frac{1}{N^{\varepsilon_2}} \implies F_{\mathrm{Q}}\left[\psi_{[l]}^N\right] \sim N^{2-\varepsilon_1-2\varepsilon_2}}$$

Note that then in asymptotic N limit $p \to 0$, so we deal with **fully depolarised state** !!!

Noise increases with N, but slowly enough !!!

- In order to attain "*exactly the HL*" ($1/N^2$) as $N\to\infty$, both the ***relative size*** (R_{LEB}) and the ***amount*** (E_G) of entanglement *cannot* be vanishing asymptotically with N.
- In order to attain "*almost the HL*" ($1/N^{2-\varepsilon}$ for any $\varepsilon > 0$) as $N\to\infty$, both the ***relative size*** (R_{LEB}) and the ***amount*** (E_G) of entanglement *may* be vanishing asymptotically with N.

Quantum Metrology with Random States

WHAT DO WE MEAN BY "RANDOM STATES"?

Isospectral quantum states – density matrices with **fixed spectrum**:

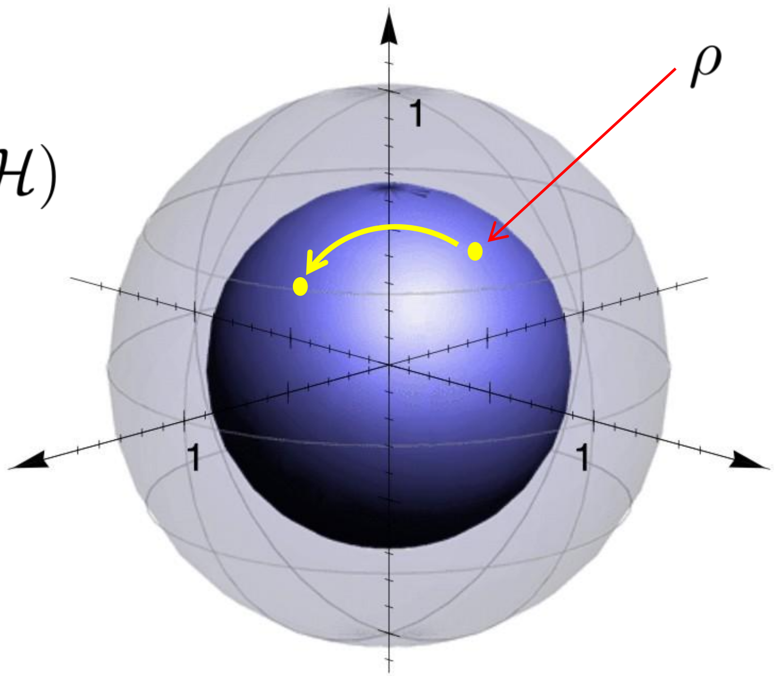
$$\{p_1, p_2, \ldots, p_d\} \implies \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \implies \rho_U = U\rho U^\dagger$$

States generated by the **unitary rotations**:

$$U \in \mathrm{SU}(\mathcal{H})$$

chosen **randomly** according to
the **uniform normalized (Haar) measure** defined for the $\mathrm{SU}(\mathcal{H})$ group: $\mu(\mathcal{H})$

$$U \sim \mu(\mathcal{H})$$



LIPSCHITZ CONTINUITY OF QFI

Lipschitz-continuous function:

$$f : X \to \mathbb{R}$$

fluctuation of the function

Lipschitz Constant

$$\frac{|f(X_1) - f(X_2)|}{\mathcal{D}(X_1, X_2)} \leq L$$

distance between elements

We have thus just proved Lipschitz continuity of QFI on states !!!!:

$$F_{\mathrm{Q}}(\hat{H}) : \rho^N \to \mathbb{R}$$

$$(\text{for fixed } \hat{H} = \sum_n \hat{h}^{(n)})$$

$$\frac{|F_{\mathrm{Q}}[\rho^N] - F_{\mathrm{Q}}[\sigma^N]|}{\mathcal{D}_{\mathrm{B}}(\rho^N, \sigma^N)} \leq \xi N^2$$

Proof...?

Ok, ok.... but we need Lipschitz continuity on the unitary group $\mathrm{SU}(\mathcal{H})$:

$$F_{\mathrm{Q}}(\hat{H}, \rho) : U \to \mathbb{R}$$

$$(\text{remember } \rho_U = U\rho U^\dagger)$$

$$\frac{|F_{\mathrm{Q}}(U_1) - F_{\mathrm{Q}}(U_2)|}{\mathcal{D}(U_1, U_2)} \leq \tilde{L}$$

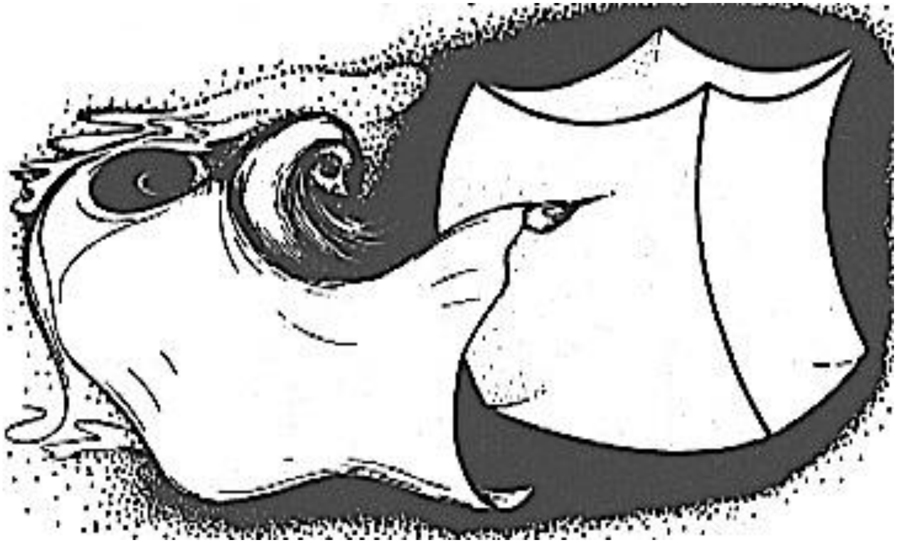
geodesic distance

QFI non-linear (SLD). Need diff-geometry...

GREAT..... BUT WHY ALL THIS LIPSCHITZ BUSINESS?

Concentration of Measure Phenomenon

Functions on high-dimensional spaces typically attain values close to their averages.



Applications of concentration of measure in quantum information:

- **Foundations of statistical mechanics** [*Popescu et al 2005*], [*Goldstein et al 2005*]
- **Hasting's disproof of additivity conjecture** [*Hastings 2009*]
- **Typical properties of entanglement for multiparticle system** [*Hayden et al 2005*]

Andreas Winter: "One of these facts of life that you just need to accept…"

Concentration of Measure Phenomenon

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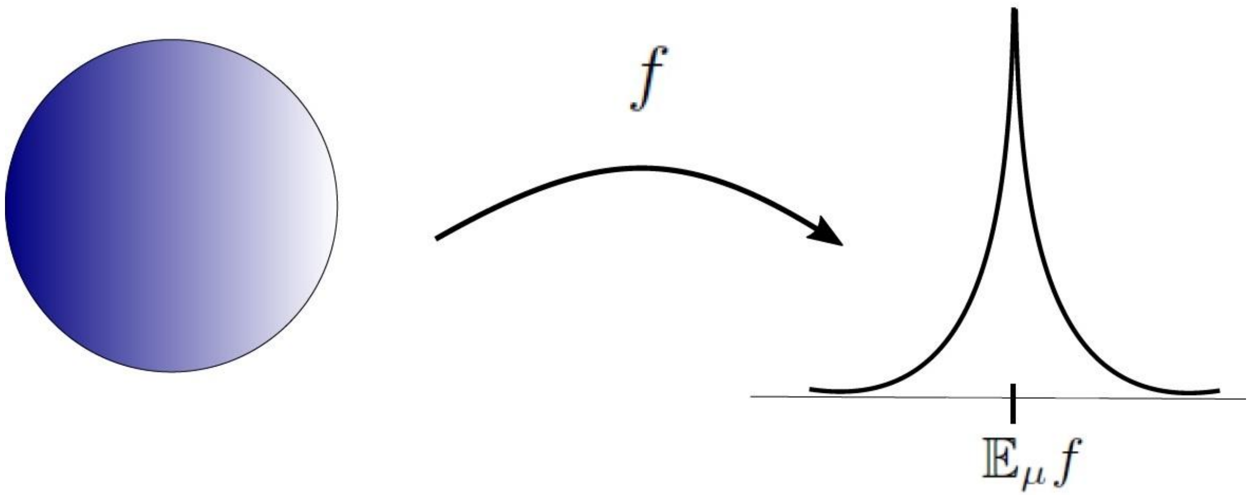


Concentration of measure on $\mathrm{SU}(\mathcal{H})$

Let $f : \mathrm{SU}(\mathcal{H}) \to \mathbb{R}$ be a function on $\mathrm{SU}(\mathcal{H})$ with the mean $\mathbb{E}_\mu f$ and Lipschitz constant[a] L. Then, the following inequality holds

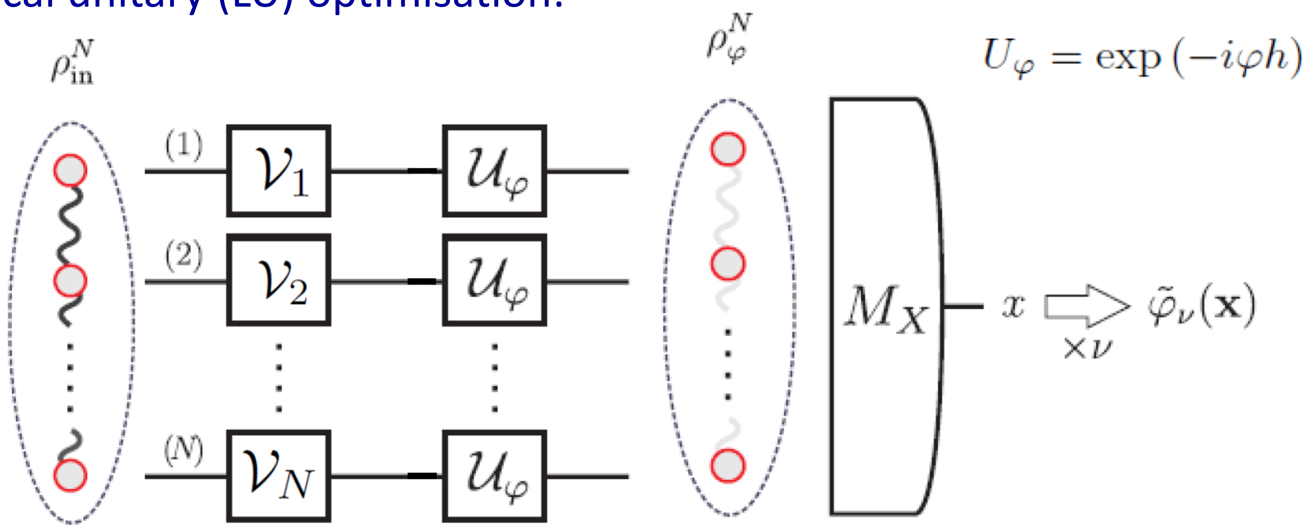
$$\mu\left(\{U \in \mathrm{SU}(\mathcal{H}) \mid |f(U) - \mathbb{E}_\mu f| \geq \epsilon\}\right) \leq 2\exp\left(-\frac{D\epsilon^2}{4L^2}\right),$$

where: μ is the Haar measure on $\mathrm{SU}(\mathcal{H})$ and $D = \dim\mathcal{H}$.

[a] With respect to the geodesic distance

TYPICAL QFI FOR DISTINGUISHABLE PARTICLES

Allow for local unitary (LU) optimisation:



LU-optimised QFI: $$F_{\text{Q}}^{\text{LU}}\left[\rho^N\right] = \sup_{V\in\text{LU}} F_{\text{Q}}\left[V\rho^N V^\dagger, H\right]$$

Result: Most random states are not useful for metrology

Fix a single-particle Hamiltonian h, local dimension d and a state ρ_N on $\mathcal{H}_N$. Let $F_{\text{Q}}^{\text{LU}}(U) = F_{\text{Q}}^{\text{LU}}\left[U\rho_N U^\dagger\right]$, then

$$\Pr_{U\sim\mu(\mathcal{H}_N)}\left(F_{\text{Q}}^{\text{LU}}(U) \notin \Theta(N)\right) \leq \exp\left(-\Theta\left(\frac{d^N}{N^2}\right)\right)$$

Typical QFI for distinguishable particles

Sketch of the proof:

- Lipschitz constant with LU-optimisation of $F_{\mathrm{Q}}^{\mathrm{LU}}\left[U\rho^{N}U^{\dagger}\right]$ follows from **Lipschitz continuity** of $F_{\mathrm{Q}}^{\mathrm{LU}}\left[\rho^{N}\right]$.
- Average value of $F_{\mathrm{Q}}^{\mathrm{LU}}\left[U\rho^{N}U^{\dagger}\right]$ can be upper-bounded using results from **typicality of entanglement** [Hayden et al. 2005]:

$$\mathbb{E}_{\mu}F_{\mathrm{Q}}^{\mathrm{LU}}\left[U\rho^{N}U^{\dagger}\right] \leq 4N\left(1+\frac{(N-1)}{\sqrt{d^{N}}}\right).$$

Animesh Datta talk — the role of two-body reduced density matrices...

- Average value of $F_{\mathrm{Q}}\left[U\rho^{N}U^{\dagger}\right]$ (no optimization!) can be **computed explicitly** and is of order N.

Actually we obtain general formula for the average QFI and any Hilbert (sub-)space:

$$\mathbb{E}_{\mu}F_{\mathrm{Q}}\left[U\rho U^{\dagger},\hat{H}\right] := \int_{\mathrm{SU}(\mathcal{H})}\mathrm{d}\mu\, F_{\mathrm{Q}}\left[U\rho U^{\dagger},\hat{H}\right] = \frac{2\cdot\mathrm{tr}\left(\hat{H}^{2}\right)}{D^{2}-1}\sum_{i,j:\,p_i+p_j\neq 0}\frac{(p_i-p_j)^2}{p_i+p_j}.$$

TYPICAL QFI FOR SYMMETRIC STATES

Inspiration: Almost all pure symmetric (*bosonic*) qubit states $|\psi\rangle \in \mathcal{S}_N$ overcome SQL after LU optimization [Hyllus et al. 2010]

(*talk of Augusto Smerzi focussing on the fact that, on the contrary, some are nevertheless not useful*).

Result: Symmetric states typically attain the Heisenberg scaling

Fix a single-particle Hamiltonian h, local dimension d and a state ρ_N from the symmetric subspace $\mathcal{S}_N$ with eigenvalues $\{p_j\}_j$. Let $F_\mathrm{Q}(U) = F_\mathrm{Q}\left[U\,\rho^N\,U^\dagger, H\right]$, then

$$\Pr_{U\sim\mu(\mathcal{S}_N)}\left(F_\mathrm{Q}\left(U\right) \leq \mathcal{D}_\mathrm{B}\left(\rho^N, \rho_\mathrm{mix}\right)^2 \Theta\left(N^2\right)\right) \leq \exp\left(-\mathcal{D}_\mathrm{B}\left(\rho^N, \rho_\mathrm{mix}\right)^3 \Theta\left(N^{d-1}\right)\right)$$

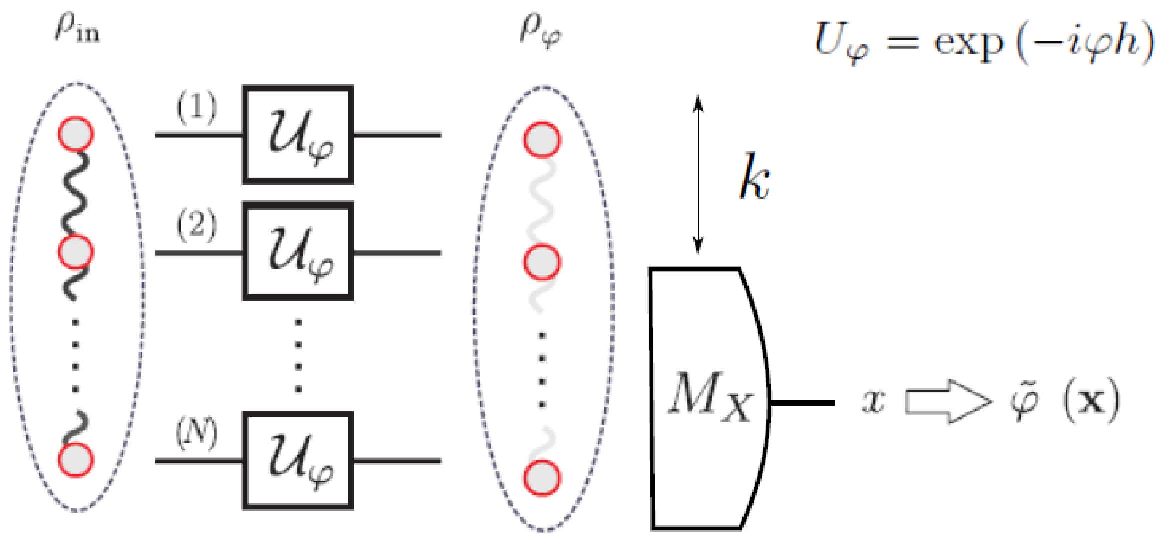
where $\mathcal{D}_\mathrm{B}\left(\rho^N, \rho_\mathrm{mix}\right)$ is the Bures distance between ρ^N and the maximally mixed state ρ_mix on $\mathcal{S}_N$.

NOISE ROBUSTNESS:

Super-classical scaling preserved as long as (noise restricted to symmetric subspace):

$$\mathcal{D}_\mathrm{B}(\rho^N, \rho_\mathrm{mix}) \gtrsim \frac{1}{N^\alpha} \quad \text{with} \quad \alpha < \frac{d-1}{3}$$

ROBUSTNESS AGAINST LOSS OF FINITE NUMBER OF PARTICLES (IN CONTRAST TO GHZ STATES)



Result: Typical robustness of QFI under finite particle losses

Fix a single particle Hamiltonian h, local dimension d, and a state ρ^N on $\mathcal{S}_N$ with eigenvalues $\{p_j\}_j$. Let $F_{Q_k}(U) = F_Q\left[\mathrm{tr}_k\left(U\,\rho^N\,U^\dagger\right), H_{N-k}\right]$, then

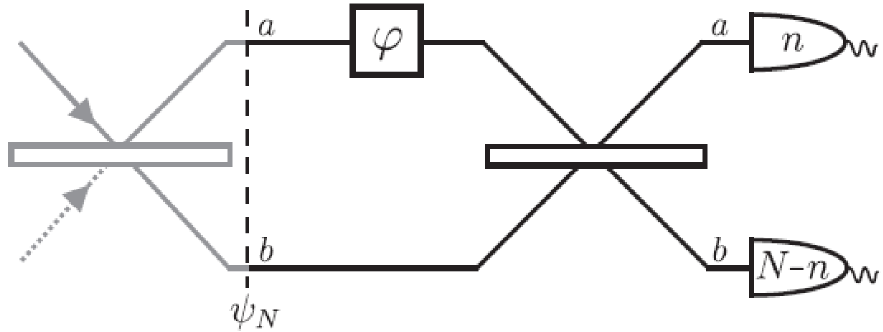
$$\Pr_{U\sim\mu(\mathcal{S}_N)}\left(F_{Q_k}(U) < \|\rho^N - \sigma_{\text{mix}}\|^2_{\text{HS}}\,\Theta\left(\frac{N^2}{k^d}\right)\right) \leq \exp\left(-\|\rho^N - \sigma_{\text{mix}}\|^4_{\text{HS}}\,\Theta\left(\frac{N^{d-1}}{k^{2d}}\right)\right),$$

where $\|\rho^N - \sigma_{\text{mix}}\|_{\text{HS}}$ is the Hilbert-Schmidt distance between ρ^N and the maximally mixed state σ_{mix} on $\mathcal{S}_N$.

- **Main idea**: use lower bound $\|[\rho, H]\|^2_{\text{HS}} \leq F[\rho, H]$.
- Setting $k = N^\alpha$ we obtain that typically $F[\rho, H] \geq N^{2-d\alpha}$.

HEISENBERG SCALING WITH FIXED MEASUREMENT SCHEME

- System N bosons in two modes (a,b) and a standard interferometric phase estimation scheme.



corresponding to the unitary ending $\psi(\varphi) = \mathrm{e}^{-\mathrm{i}\hat{J}_z\varphi}\rho\,\mathrm{e}^{\mathrm{i}\hat{J}_z\varphi}$ and measurement in the Dicke basis

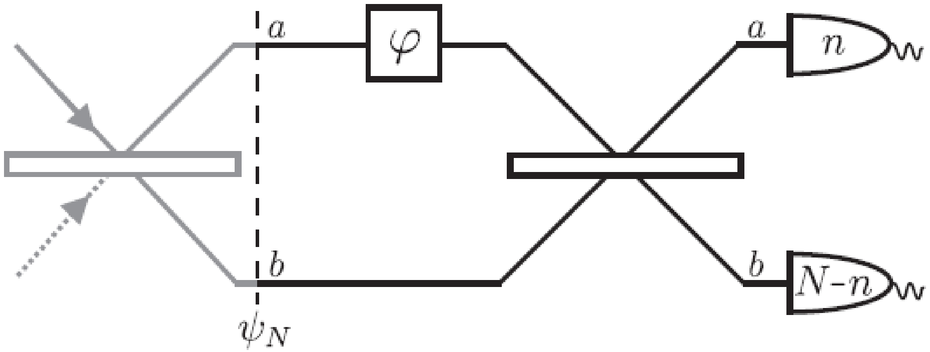
$$M_n = D_n^y \ , \ n = 0, \ldots, N \ ,$$

corresponding to $\hat{J}_y$.

- Ultimate precision of estimation of φ is quantified by the **classical FI**

$$F_{\mathrm{cl}}(\{p_{n|\varphi}(\psi)\}) = \sum_{n=0}^{N} \frac{\mathrm{tr}\Big(\mathrm{i}\,[D_n^y, \hat{J}_z]\psi(\varphi)\Big)^2}{\mathrm{tr}(D_n^y\psi(\varphi))} \ .$$

Heisenberg Scaling with Fixed Measurement Scheme



Typicality of HL in the simple interferometric setup

Let ψ_N be a fixed pure state on $\mathcal{S}_N$ and $p_{n|\varphi}(U\psi_N U^\dagger)$ be the probability of an outcome n interferometric setup above for the phase value φ and the interferometer state $U\psi_N U^\dagger$. Let $F_{\text{cl}}(U,\varphi) = F_{\text{cl}}(\{p_{n|\varphi}(U\psi_N U^\dagger)\})$, then

$$\Pr_{U\sim\mu(\mathcal{S}_N)}\left(F_{\text{cl}}(U,\varphi) \leq \Theta\left(N^2\right)\right) \leq \exp\left(-\Theta(N)\right) \ .$$

Remark: It is possible to prove a **stronger** statement:

$$\Pr_{U\sim\mu(\mathcal{S}_N)}\left(\exists_{\varphi\in[0,2\pi]}\, F_{\text{cl}}(U,\varphi) \leq \Theta\left(N^2\right)\right) \leq \exp\left(-\Theta(N)\right) \ .$$

This Means Random states make the exact phase-value problem Irrelevant!!!!

(*Michal Jachura talk...*)

Simulating Random symmetric States

- Can states mimicking the properties of Haar-random states on $\mathcal{S}_N$ be **generated efficiently**?
- **Known result**: [F. Brandao, A. Harrow and M. Horodecki 2012] Sufficiently long random circuits formed from the set of gates universal in $\mathcal{H}$ give approximate t-designs.
- Our strategy: supplement gates universal for linear optics to get universality on $\mathcal{S}_N$ for $d = 2$ modes.

Construction of the universal set of gates in $\mathcal{S}_N$:

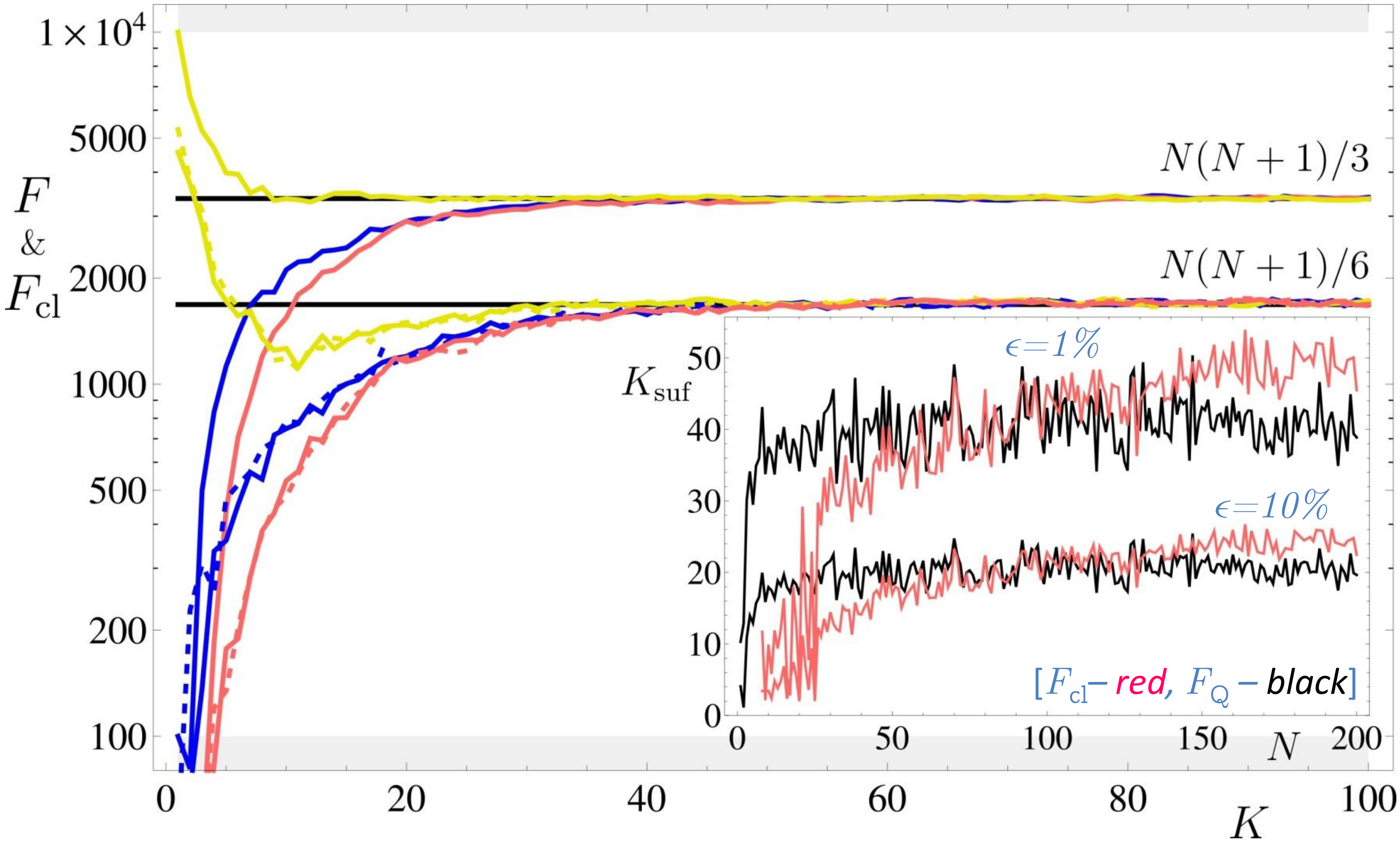
- Take single qubit gates generating $\mathrm{SU}(2)$ [Sarnak 1986]

$$V_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}, V_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix},$$
$$V_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix},$$

  and lift them to linear optics on $\mathcal{S}_N$ via $\hat{V}_i = V_i^{\otimes N}$.
- Supplement this set of gates by cross-Kerr like transformation $\hat{V}_{\mathrm{K}} = \exp\left(-i\frac{\pi}{3}\hat{n}_a\hat{n}_b\right)$.

SIMULATING RANDOM SYMMETRIC STATES

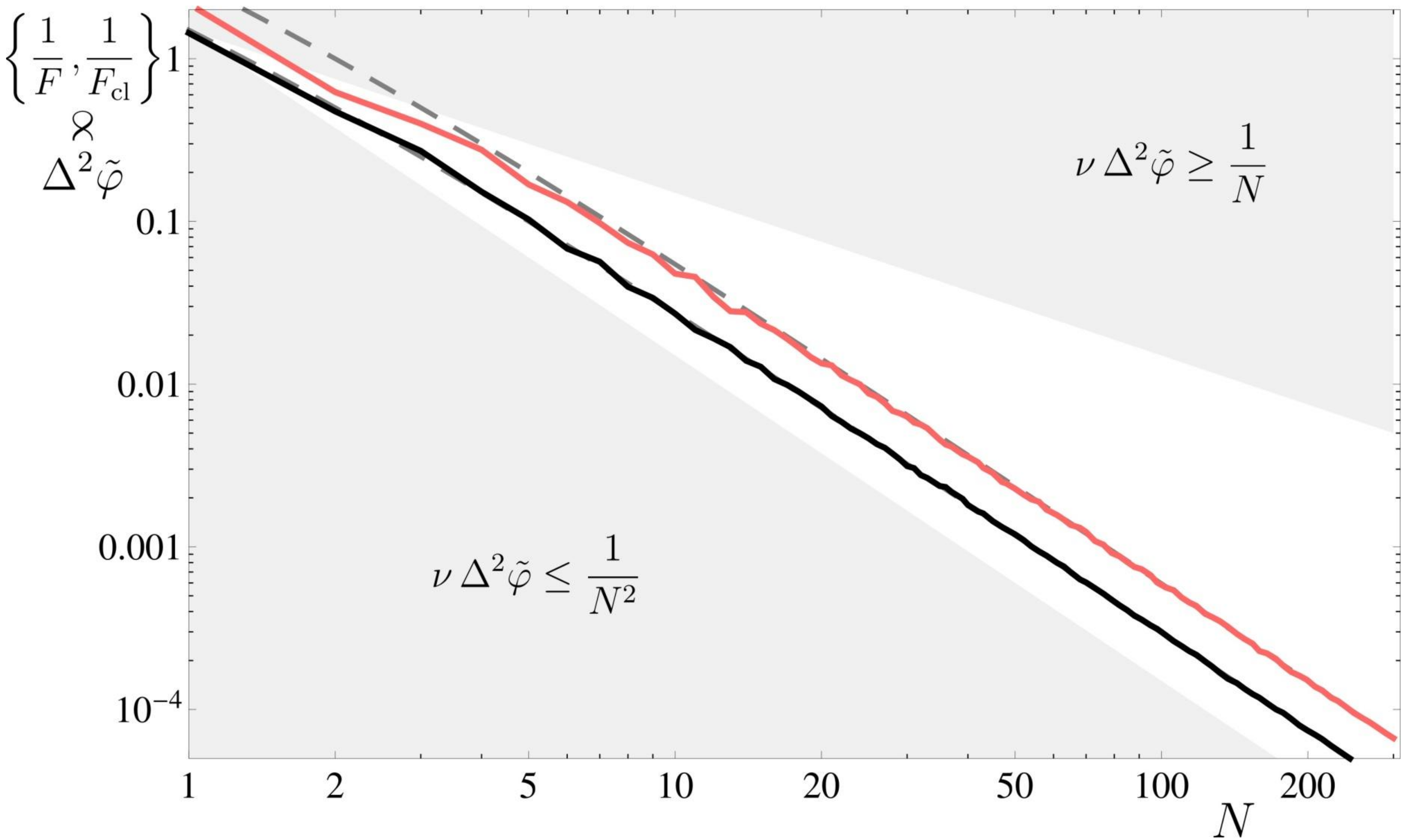
Quick (with circuit depth) saturability of the averaged QFI and classical QFI:



(N = 100, number of independent realizations = 150)

SIMULATING RANDOM SYMMETRIC STATES

Attainable precision with generated random pure symmetric (bosonic) states:



(for sufficient circuit depth and number of realizations, $F_{\rm cl}$ – *red*, $F_{\rm Q}$ – *black*)

CONCLUSIONS

arXiv.org > quant-ph > arXiv:1506.08837

1. Connection between **geometry of quantum states** and their **metrological properties**.
2. **Continuity of Quantum Fisher Information** (QFI) for unitary encoding.
3. Natural link (geometric) between the **QFI** and the **Geometric Measure of Entanglement** (E_{G}).
4. **Non-vanishing entanglement properties** are necessary to attain **the exact Heisenberg Limit**.
5. States with asymptotically **vanishing entanglement properties** can yield **precision scaling arbitrary close to the Heisenberg Limit**.

arXiv.org > quant-ph > arXiv:1602.05407

1. Random states of N **distinguishable** particles (*qudits*) are **useless** for quantum metrology despite possessing on average **high entanglement**, $E_{\mathrm{G}} \approx 1$, and even when allowing for **LU optimisation**.
2. Random states of N **symmetric** particles (*d-mode bosons*) typically **achieve the Heisenberg Limit**.
   - They are **robust against mixing** noise that ("non-exponentially") increases with system size.
   - They are **robust against particle losses** that ("sub-linearly") increase with system size.
3. Random states of N **pure symmetric particles** (bosons) typically **achieve the Heisenberg Limit with measurement fixed** to the (Mach-Zehnder) interferometric one with photon counting.
   - They can be **simulated efficiently with short random optical circuits** generated from a set of **three types of beamsplitters and a single non-linear (Kerr-type) transformation**.

THANK YOU ☺