USING ENTANGLEMENT AGAINST NOISE in quantum metrology

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Parameter estimation, possible strategies

Parallel strategies

$N$ probes

$U_\phi$

$U_\phi$

$U_\phi$

CC

Parameter estimation, possible strategies

Parallel strategies

$N$ probes

Parameter estimation, possible strategies

Parallel strategies

$N$ probes

- CC
- CQ
- QC

Parameter estimation, possible strategies

Parallel strategies

\( N \) probes

\[ \begin{array}{c}
\text{CC} \\
U_\Phi \\
U_\Phi \\
U_\Phi \\
\text{CQ} \\
U_\Phi \\
U_\Phi \\
U_\Phi \\
\text{QC} \\
U_\Phi \\
U_\Phi \\
U_\Phi \\
\text{QQ} \\
U_\Phi \\
U_\Phi \\
U_\Phi \\
\end{array} \]

Parameter estimation, possible strategies

Parallel strategies

\( N \) probes

\[
\begin{array}{c|c|c}
\text{CC} & \text{CQ} & \\
\hline
U_\varphi & U_\varphi & \\
U_\varphi & U_\varphi & \\
U_\varphi & U_\varphi & \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{QC} & \text{QQ} & \\
\hline
U_\varphi & U_\varphi & \\
U_\varphi & U_\varphi & \\
U_\varphi & U_\varphi & \\
\end{array}
\]

Sequential str.

1 probe

\begin{array}{c}
A \\
\hline
U_\varphi & U_\varphi & U_\varphi \\
\end{array}

Parameter estimation, possible strategies

Parallel strategies

$N$ probes

Sequential strategy

1 probe

Parameter estimation, possible strategies

Parallel strategies

$N$ probes

Sequential strategies

1 probe

Sequential can simulate any parallel, but takes more time!
Parallel strategies

Ancilllas are useless!!!

Sequential str.

1 probe

\[
\frac{1}{N}
\]
Sequential and entangled-parallel are equivalent in the noiseless case...

What happens in the noisy case?
Sequential and entangled-parallel are equivalent in the noiseless case...

What happens in the noisy case?

The unentangled strategy performs WORSE!!!
Sequential and entangled-parallel are equivalent in the noiseless case...

What happens in the noisy case?

The unentangled strategy performs WORSE!!!

Then you should use entanglement to achieve higher sensitivity in the presence of noise!

very surprising!
simple example: erasure noise.

simple to see: noise and unitary commute!

(a) Optimal sequential strategy, which is equivalent to a (b) sequential strategy where a (larger) erasure happens at the end, which is equivalent to a specific (c) parallel-entangled strategy where the erasure is only on the first probe, which is equivalent to a (d) parallel-entangled strategy in the presence of erasure on all probes. This last is weaker than the optimal parallel-entangled strategy, since the input state is not optimized.
General hierarchy of metrology strategies
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(i) = (ii) = (iii) = (iv)  
(i) < (ii) = (iii) = (iv)  
(i) < (ii) < (iii) \neq (iv)  
(i) \leq (ii) \leq (iii) \neq (iv)  

decoherence free,  
dephasing, erasure,  
amplitude-damping,  
general conjecture.
Fisher info for these strategies:

\[
F^{(i)} = \max_{\rho,n} F\{[\Lambda^n_\varphi(\rho)]^{\otimes N/n}\},
\]

\[
F^{(ii)} = \max_{\rho_N} F[\Lambda^{\otimes N}_\varphi(\rho_N)],
\]

\[
F^{(iii)} = \max_{\rho_M} F[\Lambda^{\otimes N}_\varphi \otimes 1^{\otimes M}(\rho_M)],
\]

\[
F^{(iv)} = \max_{\rho_M, \{U_i\}} F[U_N \Lambda_\varphi \ldots U_1 \Lambda_\varphi(\rho_M)],
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Obvious results:

\[F^{(ii)} \leq F^{(iii)} \quad F^{(iii)} \leq F^{(iv)}\]
Fisher info for these strategies:

\[ F^{(i)} = \max_{\rho,n} F\{[\Lambda_n^\varphi(\rho)]^{\otimes N/n}\}, \]

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Obvious results:

\[ F^{(ii)} \leq F^{(iii)} \quad F^{(iii)} \leq F^{(iv)} \]

Other relations?
It's difficult to calculate Fl:
use upper bounds:
It's difficult to calculate FI: use **upper bounds**:

$$\max_{\rho} F[\Lambda_\phi(\rho)] \leq 4 \min_{\{K_\phi^k\}} \| \sum_k K_\phi^k \dot{K}_\phi^k \|,$$

$$\dot{K}_\phi^k = (\partial K_\phi^k / \partial \phi)$$
It's difficult to calculate FI: use upper bounds:

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\max_{\rho} F[\Lambda_{\phi}(\rho)] \leq 4\min_{\{K^\phi_k\}} \sum_k \dot{K}^\phi_k \dot{K}^\phi_k,
\]

\[
\dot{K}^\phi_k = (\partial K^\phi_k / \partial \phi)
\]

using the parallel structure of the Kraus maps, we can prove:

\[
F^{(ii/iii)} \leq 4\min_{K^\phi_k} N\|\alpha\| + N(N - 1)\|\beta\|^2
\]

\[
F^{(iv)} \leq 4\min_{K^\phi_k} N\|\alpha\| + N(N - 1)\|\beta\|(\|\alpha\| + \|\beta\| + 1)
\]

\[
\alpha \equiv \sum_k \dot{K}^\phi_k \dot{K}^\phi_k \text{ and } \beta \equiv \sum_k \dot{K}^\phi_k \dot{K}^\phi_k
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using the parallel structure of the Kraus maps, we can prove:

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\alpha \equiv \sum_k \dot{K}^\varphi_{k} \dot{K}^\varphi_{k} \quad \text{and} \quad \beta \equiv \sum_k \dot{K}^\varphi_{k} \dot{K}^\varphi_{k}
\]

(ii) and (iii) have the same bound: are they equivalent?
It's difficult to calculate $F_1$: use upper bounds:

$$\max_{\rho} F[\Lambda_{\phi}(\rho)] \leq 4 \min_{\{K_k^{\phi}\}} \|\sum_k \dot{K}_k^{\phi} \dot{K}_k^{\phi}\|,$$

where $\dot{K}_k^{\phi} = (\partial K_k^{\phi}/\partial \phi)$.

using the parallel structure of the Kraus maps, we can prove:

$$F^{(ii/iii)} \leq 4 \min_{K_k^{\phi}} \|\alpha\| + N(N - 1)\|\beta\|^2$$

$$F^{(iv)} \leq 4 \min_{K_k^{\phi}} \|\alpha\| + N(N - 1)\|\beta\|(\|\alpha\| + \|\beta\| + 1)$$

$$\alpha \equiv \sum_k \dot{K}_k^{\phi} \dot{K}_k^{\phi} \text{ and } \beta \equiv \sum_k \dot{K}_k^{\phi} K_k^{\phi}$$

(ii) and (iii) have the same bound: are they equivalent? NO!! amplitude damping: (ii) < (iii)
It's difficult to calculate FI: use upper bounds:

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\max_{\rho} F[\Lambda_{\phi}(\rho)] \leq 4 \min_{\{K_k^\phi\}} \| \sum_k \dot{K}_k^\phi \dot{K}_k^\phi \|
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K_k^\phi = (\partial K_k^\phi / \partial \phi)
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\]

(ii) and (iii) have the same bound: are they equivalent? **NO!!** amplitude damping: (ii) < (iii)

(ii/iii) and (iv) have different bound: are they inequivalent?
It's difficult to calculate FI:
use upper bounds: \[ \max_{\rho} F[\Lambda_{\varphi}(\rho)] \leq 4\min_{\{K_k^\varphi\}} \| \sum_k \dot{K}_k^\varphi \dot{K}_k^\varphi \|, \]

using the parallel structure of the Kraus maps, we can prove:

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(ii) and (iii) have the same bound: are they equivalent?

\textbf{NO!!} \quad \text{amplitude damping: } (ii) < (iii)

(ii/iii) and (iv) have different bound: are they inequivalent?

\textbf{NO!!} \quad (\text{maybe they're equivalent! CONJECTURE!})
What about the sequential strategy (i)?
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I've already shown that it's strictly worse than (ii) for erasure,
What about the sequential strategy (i)?

I've already shown that it's strictly worse than (ii) for erasure, that's also true for dephasing.
What about the sequential strategy (i)?

I've already shown that it's strictly worse than (ii) for erasure, that's also true for dephasing (but it's not true for amplitude damping!)
What about the sequential strategy (i)?

I've already shown that it's strictly worse than (ii) for erasure, that's also true for dephasing (but it's not true for amplitude damping!)

Do we know that (i) is **always** worse or equal than (ii)?
What about the sequential strategy (i)?

I've already shown that it's strictly worse than (ii) for erasure, that's also true for dephasing (but it's not true for amplitude damping!)

Do we know that (i) is **always** worse or equal than (ii)?

**NO!!!** (conjecture)
Open question!

is entanglement at the measurement stage useful!??

(it's useless in the noiseless case!)

[recent work by Kavan Modi?]
Summary:
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Take two strategies (an unentangled and an entangled one) that are equivalent without noise. Which one is better when you add noise?
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The entangled one!!
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The entangled one!! WHAT!?!
Summary:

Take two strategies (an unentangled and an entangled one) that are equivalent without noise. Which one is better when you add noise? The entangled one!! WHAT!?

• the role of ancillas in q metrology?
Take two strategies (an unentangled and an entangled one) that are equivalent without noise. Which one is better when you add noise?

The entangled one!! WHAT!?

• the role of ancillas in q metrology?

• Useless in the noiseless case :-(

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Summary:

Take two strategies (an unentangled and an entangled one) that are equivalent without noise. Which one is better when you add noise?

The entangled one!! WHAT!?

- the role of ancillas in q metrology?
  - Useless in the noiseless case :-(
  - Useful in the noisy case :-)

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Entangled protocols may be more robust to noise than unentangled ones which are equivalent without noise!
Entanglement and Complementarity

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What I'm going to talk about

We always say that entangled states are more correlated... WHAT DOES IT MEAN exactly?
What I'm going to talk about

We always say that entangled states are more correlated... WHAT DOES IT MEAN exactly?

they have more correlations among complementary observables than separable ones
Usual approaches to study entanglement

- Non locality
- LOCC (?!?!)
- Bell inequality violations
- Enhanced precision in measurements
- etc.
Here: we use correlations among two (or more) COMPLEMENTARY PROPERTIES.

different way to think about entanglement, as correlations among complementary properties.
Remember: Complementary properties.
Remember: Complementary properties.

Two observables: the knowledge of one gives no knowledge of the other

\[ A = \sum_a f(a) |a \rangle \langle a| \]
\[ C = \sum_c g(c) |c \rangle \langle c| \]

\[ |\langle a|c \rangle|^2 = \frac{1}{d} \]
simplest example:
simplest example:

\[
\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|++\rangle + |--\rangle}{\sqrt{2}}
\]

Maximally entangled state: perfect correlation BOTH on 0/1 and on +/-
simplest example:

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\]

Maximally entangled state: perfect correlation BOTH on 0/1 and on +/-

\[
\frac{|00\rangle\langle00| + |11\rangle\langle11|)}{2} = \\
\frac{|+\rangle\langle+| + |--\rangle\langle--|)}{2} \otimes \frac{|+\rangle\langle+| + |--\rangle\langle--|)}{2}
\]

separable state: perfect correlation for 0/1, no correlation for +/-
Simple experiment

• On system 1 measure either A or C
• On system 2 measure either B or D
• Calculate correlations $A-B$ and $C-D$
How to measure correlation?
How to measure correlation?

- Mutual information

\[ I_{AB} = H(A) + H(B) - H(A, B) \]
How to measure correlation?

- Mutual information

\[ I_{AB} = H(A) + H(B) - H(A, B) \]

- Pearson correlation coefficient

\[ C_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B} \quad \text{or anticorrelation} \]
Use these to measure correlations among

2 complementary properties

\[ A \otimes B \quad \text{complem to} \quad C \otimes D \]

of 2 systems

possible meas.

source

possible meas.
Some results...
Start with mutual information

\[ I_{AB} = H(A) + H(B) - H(A, B) \]

“total” correlation given by the sum

\[ I_{AB} + I_{CD} \]
The system state is maximally entangled iff perfect correlation on both $A-B$ and $C-D$.
The system state is maximally entangled iff perfect correlation on both \( A-B \) and \( C-D \)

\[
I_{AB} + I_{CD} = 2 \log d
\]

(for some observ \( ABCD \))

\[
\Leftrightarrow \quad |\Psi_{12}\rangle \text{ maximally entangled}
\]
The system state is maximally entangled iff perfect correlation on both $A-B$ and $C-D$.

\[ I_{AB} \leq \log_2 d \]

\[ I_{CD} \leq \log_2 d \]

\[ I_{AB} + I_{CD} = 2 \log d \]

(for some observ $ABCD$)

$|\Psi_{12}\rangle$ maximally entangled
The system state is **entangled** if correlations on **both** $A-B$ and $C-D$ are large enough.
The system state is entangled if correlations on both $A-B$ and $C-D$ are large enough.

\[ I_{AB} + I_{CD} > \log d \]
The system state is entangled if correlations on both \( A-B \) and \( C-D \) are large enough:

\[
I_{AB} \leq \log_2 d \\
I_{CD} \leq \log_2 d
\]

Thus,

\[
I_{AB} + I_{CD} > \log d \implies \rho_{12} \text{ ent}
\]
The system state is \textbf{entangled} if correlations on both $A-B$ and $C-D$ are large enough.

\[ I_{AB} + I_{CD} > \log d \]

Can the bound be made \textbf{tighter}?
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

$$I_{AB} + I_{CD} > \log d$$

Can the bound be made **tighter**?

**NO!!**
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

\[ I_{AB} + I_{CD} > \log d \quad \text{ent} \]

Can the bound be made **tighter**?

**NO!!**

the **separable** state

\[ \frac{1}{2}(\ket{00}\bra{00} + \ket{11}\bra{11}) \]

saturates it:

\[ I_{AB} + I_{CD} = \log d \]
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

is the converse true?
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

is the converse true?

**No!!**
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

Is the converse true? **NO!!**

$$|\psi_\epsilon\rangle = \epsilon |00\rangle + \sqrt{1 - \epsilon^2} |11\rangle$$

Is entangled but has negligible mutual info for $\epsilon \rightarrow 0$.
Another measure of correlation...
Pearson correlation coefficient

\[ C_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B} \]

\[ |C_{AB}| = 1 \Rightarrow \text{perfect correlation or anticorrelation} \]
Pearson correlation coefficient

$$C_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B}$$

$$|C_{AB}| = 1 \Rightarrow \text{perfect correlation or anticorrelation}$$

it can be complex for quantum expectation values
Pearson correlation coefficient

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| \[ C_{AB} \] | = 1 \Rightarrow \text{perfect correlation or anticorrelation}

it can be \textbf{complex} for quantum expectation values

... but its modulus is still \( \leq \left| 1 \right| : \)
Pearson correlation coefficient

\[ C_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B} \]

| \[ |C_{AB}| = 1 \Rightarrow \] |

perfect correlation or anticorrelation

it can be complex for quantum expectation values

not a problem for us: \( A \) and \( B \) commute, so it's REAL

\[ A \otimes B = A \otimes 1 + 1 \otimes B \]
Total correlation: again use the sum

$$|C_{AB}| + |C_{CD}|$$
The system state is **maximally entangled** iff perfect correlation on both $A-B$ and $C-D$.
The system state is maximally entangled iff perfect correlation on both $A-B$ and $C-D$

True also using Pearson! (for linear observables: Pearson measures only linear correl)
The system state is maximally entangled iff perfect correlation on both A-B and C-D.

True also using Pearson! (for linear observables: Pearson measures only linear correl)

\[ |C_{AB}| + |C_{CD}| = 2 \] (for some observ A BCD)

\[ \iff |\Psi_{12}\rangle \] maximally entangled
The system state is maximally entangled if\,\,\,iff \,\,\,perfect correlation on both $A-B$ and $C-D$ are true also using Pearson! (for linear observables: Pearson measures only linear correl)
The system state is entangled if correlations on both $A-B$ and $C-D$ are large enough?
The system state is entangled if correlations on both $A-B$ and $C-D$ are large enough?

**CONJECTURE**: we don't know if it's true also using Pearson!
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough?

**CONJECTURE**: we don't know if it's true also using Pearson!

\[ |C_{AB}| + |C_{CD}| > 1 \Rightarrow \rho_{12} \text{ ent} \]

(for some observ $ABCD$)
The system state is entangled if correlations on both $A-B$ and $C-D$ are large enough.

**CONJECTURE**: we don't know if it's true also using Pearson!

$$|C_{AB}| \leq 1$$

$$|C_{CD}| \leq 1$$

$$|C_{AB}| + |C_{CD}| > 1 \implies \rho_{12} \text{ ent}$$

(for some observ $ABCD$)
Conjecture: $|C_{AB}| + |C_{CD}| > 1 \implies \text{state is ent.}$

Again, the inequality is tight:
Conjecture: $|C_{AB}| + |C_{CD}| > 1 \Rightarrow \text{state is ent.}$

Again, the inequality is tight:

separable state $\frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$

$|C_{AB}| + |C_{CD}| = 1$
Conjecture: $|C_{AB}| + |C_{CD}| > 1 \Rightarrow \text{state is ent.}$

Again, the inequality is tight:

Separable state $\frac{|00\rangle\langle00| + |11\rangle\langle11|}{2}$

$|C_{AB}| + |C_{CD}| = 1$

(perfect correl on one basis, no correl on the complement)
Is the Pearson correlation weaker than the mutual info?
Is the Pearson correlation weaker than the mutual info?

NO!!

- only linear correlations
- all correlations
Is the Pearson correlation weaker than the mutual info?

**NO!!**

\[
|\psi_\epsilon\rangle = \epsilon |00\rangle + \sqrt{1 - \epsilon^2} |11\rangle
\]

Has negligible mutual info for \(\epsilon \to 0\).
Is the Pearson correlation weaker than the mutual info?

**NO!!**

\[ |\psi_\epsilon\rangle = \epsilon |00\rangle + \sqrt{1 - \epsilon^2} |11\rangle \]

Has negligible mutual info for \( \epsilon \to 0 \)

but Pearson correlation always \( > 1! \)
Simple criterion for entanglement detection!!

Just measure two complementary properties. Are the correlations greater than perfect correlation on one?

⇒ The state is entangled!

Simple to measure and simple to optimize.

Unfortunately: not very effective in finding entanglement in random states
• Entanglement as correlation among complementary observables

• Using different measures of correlation:
  • Mutual info
  • Pearson correlation

• Some theorems and some conjectures
The most correlated states are entangled but ent states are not the most correlated.