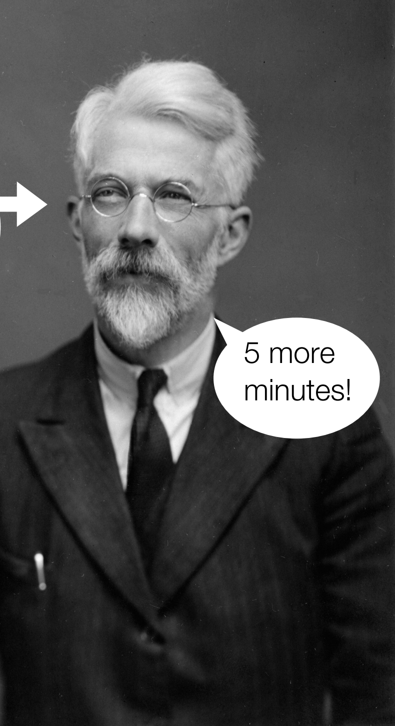
Quantum metrology with full and fast quantum control

Pavel Sekatski, Michalis Skotiniotis, Janek Kolodynski, Wolfgang Dür

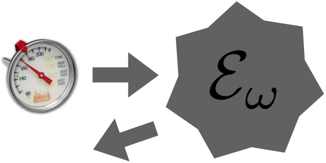
0.Background

Metrology


200
180
160
140
CHIX
BEEF WELL
BEEF MED
BEEF RARE


5 more minutes!

Metrology



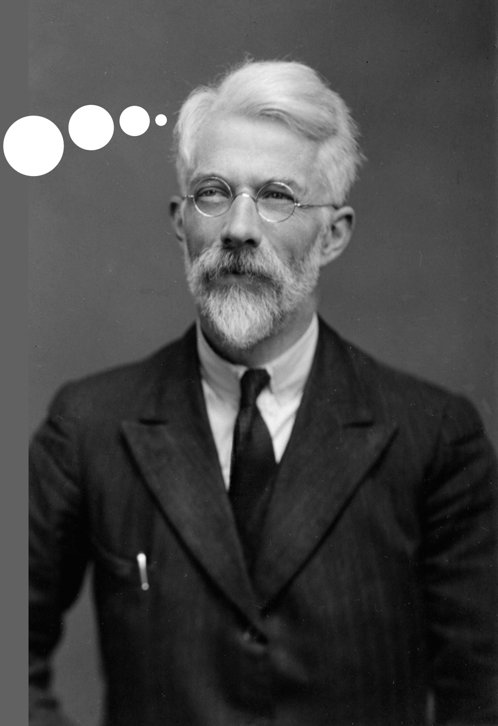
$\hat{X}$ from $p(x|\omega)$

Fisher information:

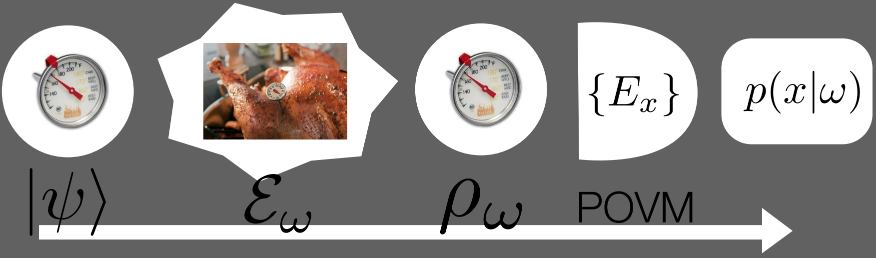
$$\mathcal{F}_\omega = \sum_x \frac{\dot{p}(x|\omega)^2}{p(x|\omega)}$$

Cramer-Rao bound:

$$\delta\omega^2 \geq \frac{1}{\nu \mathcal{F}_\omega}$$

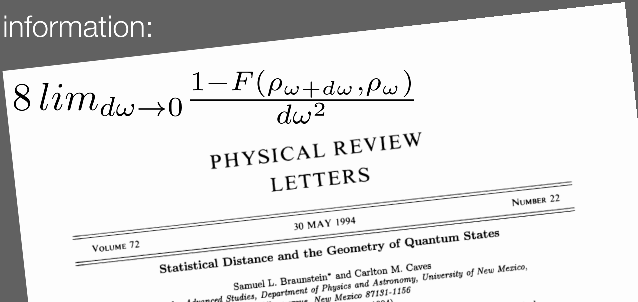


Quantum Metrology



Quantum Fisher information:

$$\mathcal{F}(\rho_\omega) = 8\, lim_{d\omega \to 0} \frac{1 - F(\rho_{\omega + d\omega}, \rho_\omega)}{d\omega^2}$$



PHYSICAL REVIEW LETTERS

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Statistical Distance and the Geometry of Quantum States

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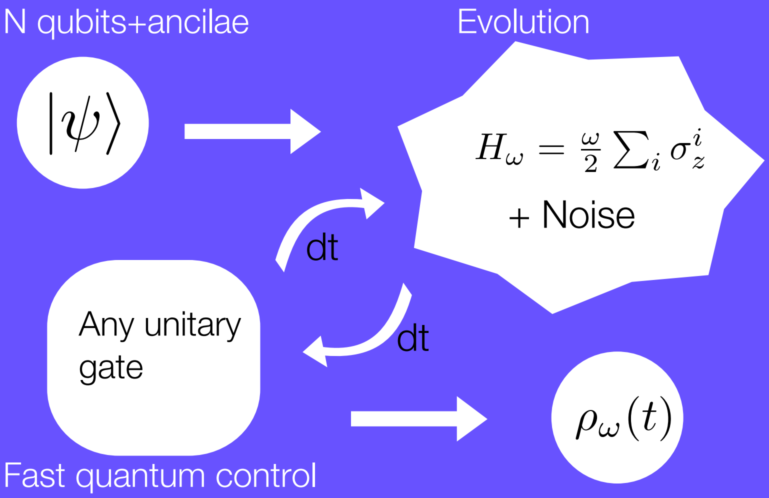
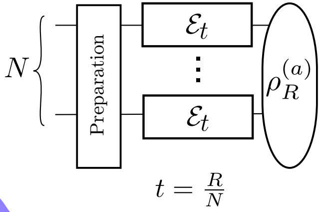
1.Introduction

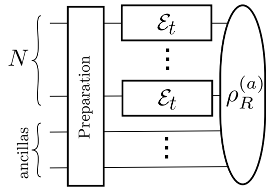
Full and fast quantum control



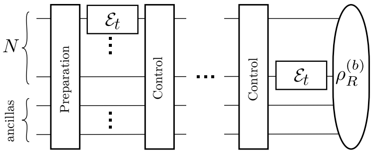
Parallel strategy



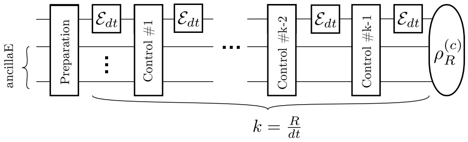
Parallel with ancilae



Sequential with full control



Sequential with full and fast control



Noise description

(how fast is fast?)

ultrafast → *superduperfast*

Lindbladian description

Master equation

$$\dot{\rho} = i[H_\omega, \rho] + \sum_i \mathcal{L}_i(\rho)$$

(time homogenous)

Hamiltonian description

Lindbladian noise

$$\mathcal{L}(\rho) := \frac{1}{2} \sum_{\mu,\nu=0}^{3} \mathsf{L}_{\mu\nu} \left([\sigma_\mu \rho, \sigma_\nu] + [\sigma_\mu, \rho \sigma_\nu] \right)$$

Rank-one noise

$$\bar{\mathsf{L}}^{1\mathrm{G}}_{\mathbf{r}} := \mathbf{r}\,\mathbf{r}^\dagger$$

Rank-one Pauli (dephasing)

$$\mathrm{Re}(\mathbf{r}) \times \mathrm{Im}(\mathbf{r}) = 0$$

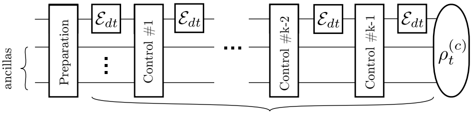
Amplitude damping

$$|\mathrm{Re}(\mathbf{r})| = |\mathrm{Im}(\mathbf{r})| \text{ and } \mathrm{Re}(\mathbf{r})^T \mathrm{Im}(\mathbf{r}) = 0$$

Rank-two Pauli noise

$$\bar{\mathsf{L}}^{2\mathrm{P}}_{\mathbf{\Omega}} := \frac{1}{2} R_{\mathbf{\Omega}}^T \begin{pmatrix} \gamma_1 & & \\ & \gamma_2 & \\ & & 0 \end{pmatrix} R_{\mathbf{\Omega}}$$

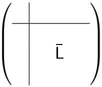
A few remarks



$k = \frac{t}{dt}$

$$\dot{\rho} = -\frac{\omega}{2} i[\sigma_3, \rho] + \mathcal{L}(\rho)$$

L =



$$\mathcal{L}(\rho) := \frac{1}{2} \sum_{\mu,\nu=0}^{3} \mathsf{L}_{\mu\nu} \left([\sigma_\mu \rho, \sigma_\nu] + [\sigma_\mu, \rho \sigma_\nu] \right)$$

1. Unitary noise can be removed

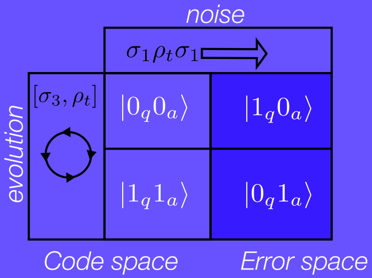
2. Perpendicular Pauli noise can be corrected:

$$\rho_{t+dt} = \rho_t(1 - \tfrac{\gamma}{2} dt) - i dt \, \tfrac{\omega}{2} [\sigma_3, \rho_t] + dt \tfrac{\gamma}{2} \sigma_1 \rho_t \sigma_1 + O(dt^2)$$

Encoding:

$$|0\rangle \to |0_q 0_a\rangle$$

$$|1\rangle \to |1_q 1_a\rangle$$



2.Quntum metrology with full and fast control

Bounding the QFI of a channel

Uhlmann-Josza Fidelity

$$F(\mathcal{E}_{\omega_1}(|\psi\rangle),\mathcal{E}_{\omega_2}(|\psi\rangle)) \geq F(\sum_k K_k^{\omega_1}|\psi\rangle\langle\psi|K_k^{\omega_1\dagger}, \sum_k \tilde{K}_k^{\omega_2}|\psi\rangle\langle\psi|\tilde{K}_k^{\omega_2\dagger}) \geq$$

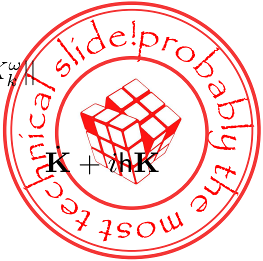
$$\sum_k \sqrt{p_k\tilde{p}_k}\, F\left(\frac{K_k^{\omega_1}|\psi\rangle}{\sqrt{p_k}}, \frac{\tilde{K}_k^{\omega_2}|\psi\rangle}{\sqrt{\tilde{p}_k}}\right) \geq \sum_k |\langle\psi|K_k^{\omega_1\dagger}\tilde{K}_k^{\omega_2}|\psi\rangle| \geq |\langle\psi|\sum_k K_k^{\omega_1\dagger}\tilde{K}_k^{\omega_2}|\psi\rangle|$$

Quantum Fisher Information

$$\mathcal{F}_{\mathcal{E}(\omega)} \leq 4\,|\langle\psi|\sum_k \dot{K}_k^{\omega\dagger}\dot{K}_k^{\omega}|\psi\rangle| \leq 4\,||\sum_k \dot{K}_k^{\omega\dagger}\dot{K}_k^{\omega}||$$

Explicit dependence on the parameter

$$\dot{\mathbf{K}} + i h \mathbf{K}$$

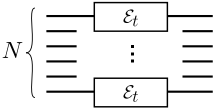


Bound for a parallel channel

$\mathcal{F}_{\mathcal{E}(\omega)} \leq 4||\sum_k \dot{K}_k^{\omega\dagger}\dot{K}_k^{\omega}||$

Kraus representation

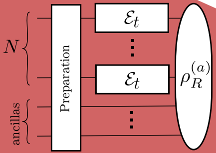
$$K_{\mathbf{k}}^{\omega} = K_{k_1}^{\omega} \otimes ... \otimes K_{k_N}^{\omega}$$



Quantum Fisher Information

$$\mathcal{F}_{\mathcal{E}(\omega)} \leq 4N\underbrace{||\sum_k \dot{K}_k^{\omega\dagger}\dot{K}_k^{\omega}||}_{||\alpha(t)||} + 4N(N-1)\underbrace{||\sum_k \dot{K}_k^{\omega\dagger}K_k^{\omega}||^2}_{||\beta(t)||^2} \quad = 0$$

Linear scaling in N!



Sequential channel with control

$\mathcal{F}_{\mathcal{E}(\omega)} \leq 4||\sum_k \dot{K}_k^{\omega\dagger}\dot{K}_k^{\omega}||$

Kraus representation

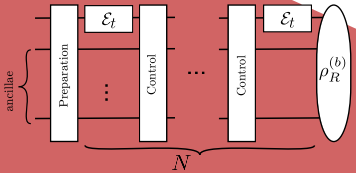
$$K_{\mathbf{k}}^{\omega} = K_{k_N}^{\omega} U_N \ldots K_{k_1}^{\omega} U_1^{\omega}$$

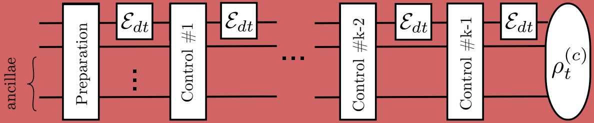
Quantum Fisher Information

$$\mathcal{F}_{\mathcal{E}(\omega)} \quad \leq \quad 4N\,||\alpha(t)|| + 4N(N-1)||\beta(t)||\left(||\beta(t)|| + \frac{||\alpha(t)||}{x} + x\right)$$

$||\beta(t)|| = 0$

Linear scaling in N!





$$\mathcal{F}_{\mathcal{E}(\omega)} \leq 4k\,||\alpha(t)|| + 4k(k-1)||\beta(t)||\Big(||\beta(t)|| + \frac{||\alpha(t)||}{x} + x\Big)$$

Take the limit $dt \to 0 \quad k = \frac{t}{dt} \to \infty$

$$4\frac{t}{dt}\,||\alpha(dt)|| + 4\frac{t^2}{dt^2}||\beta(dt)||\Big(||\beta(dt)|| + \frac{||\alpha(dt)||}{\sqrt{dt}} + \sqrt{dt}\Big)$$

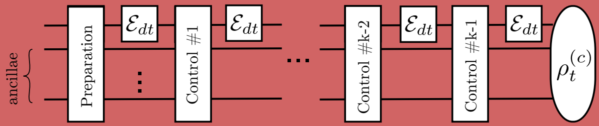
No need to solve the master equation

$$\mathcal{E}_{dt}(\omega) = \mathcal{E}^{(0)} + dt\,\mathcal{E}^{(2)}(\omega) + \ldots$$

trivial *given by the Lindblad matrix L*

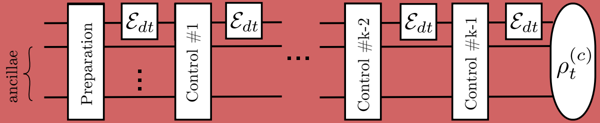
$$\mathbf{K} = \mathbf{K}^{(0)} + \sqrt{dt}\,\mathbf{K}^{(1)} + dt\,\mathbf{K}^{(1)} + \ldots \qquad \alpha(dt) = \alpha^{(0)} + \sqrt{dt}\,\alpha^{(1)} + dt\,\alpha^{(2)} + \ldots$$

$$\mathsf{h} = \mathsf{h}^{(0)} + \sqrt{dt}\,\mathsf{h}^{(1)} + dt\,\mathsf{h}^{(2)} + \ldots \qquad \beta(dt) = \beta^{(0)} + \sqrt{dt}\,\beta^{(1)} + dt\,\beta^{(2)} + \ldots$$



If there exist **h** s.t. $\beta^{(2)} = 0$ then $\mathcal{F}(\rho_t(\omega)) \leq 4t\,||\alpha^{(2)}||$

$$\beta^{(2)} = i\,\left(\frac{-1}{2}\sigma_3 + \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}^{\dagger} \left(\begin{array}{c|c} h_{00}^{(2)} & \mathbf{h}^{(1)\dagger}\bar{\mathsf{L}}^{1/2} \\ \hline \bar{\mathsf{L}}^{1/2}\mathbf{h}^{(1)} & \bar{\mathsf{L}}^{1/2}\mathsf{H}^{(0)}\bar{\mathsf{L}}^{1/2} \end{array}\right) \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}\right)$$



Result 1: For all Lindbladians except the the rank-one Pauli noise the QFI is upper-bounded by:

$$\mathcal{F}_{\mathcal{L}} \leq 4\,\alpha_{\mathcal{L}} t$$

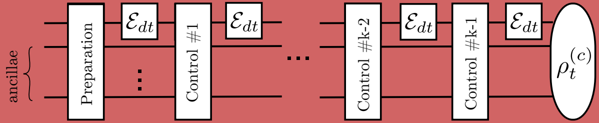
$$\bar{\mathsf{L}}_{\mathbf{r}}^{1\mathrm{G}} := \mathbf{r}\,\mathbf{r}^{\dagger}$$

$$\bar{\mathsf{L}}_{\mathbf{\Omega}}^{2\mathrm{P}} := \frac{1}{2} R_{\mathbf{\Omega}}^{T} \begin{pmatrix} \gamma_1 & & \\ & \gamma_2 & \\ & & 0 \end{pmatrix} R_{\mathbf{\Omega}}$$

$$\mathcal{F}_{\mathbf{z}}^{1\mathrm{P}} \leq \frac{t}{2\gamma}$$

$$\mathcal{F}_{\mathbf{r}}^{1\mathrm{G}} \leq \frac{t \;\; \max\, |x \pm \mathrm{i}y|^2}{4(|\mathrm{Re}(\mathbf{r})|^2 |\mathrm{Im}(\mathbf{r})|^2 - (\mathrm{Re}(\mathbf{r})^T \mathrm{Im}(\mathbf{r}))^2)}$$

$$\mathcal{F}_{\mathbf{\Omega}}^{2\mathrm{P}} \leq \frac{t}{2\gamma_1\gamma_2}\Big(c_\theta^2(\gamma_1+\gamma_2) + s_\theta^2(\gamma_1 s_\varphi^2 + \gamma_2 c_\varphi^2)\Big).$$



Result 1: For all Lindbladians except the the rank-one Pauli noise the QFI is upper-bounded by:

$$\mathcal{F}_{\mathcal{L}} \leq 4\,\alpha_{\mathcal{L}} t$$

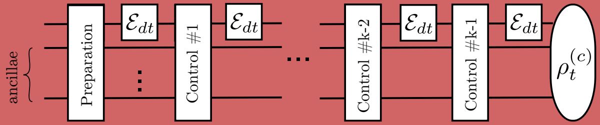
NOISE WINS

$$\mathsf{L}^{1\mathrm{G}}_{\mathbf{r}} := \mathbf{r}\,\mathbf{r}^{\dagger}$$

$$\mathcal{F}^{1\mathrm{G}}_{\mathbf{r}} \leq \frac{\ldots \max |x \pm \mathrm{i}y|^2}{4(|\mathrm{Re}(\mathbf{r})|^2 |\mathrm{Im}(\mathbf{r})|^2 - (\mathrm{Re}(\mathbf{r})^T \mathrm{Im}(\mathbf{r}))^2)}$$

$$\mathsf{L}^{2\mathrm{P}}_{\mathbf{\Omega}} := \frac{1}{2} R^{T}_{\mathbf{\Omega}} \begin{pmatrix} \gamma_1 & & \\ & \gamma_2 & \\ & & 0 \end{pmatrix} R_{\mathbf{\Omega}}$$

$$\mathcal{F}^{2\mathrm{P}}_{\mathbf{\Omega}} \leq \frac{t}{2\gamma_1\gamma_2}\Big(c_\theta^2(\gamma_1 + \gamma_2) + s_\theta^2(\gamma_1 s_\varphi^2 + \gamma_2 c_\varphi^2)\Big).$$



What about rank-one Pauli noise $\mathcal{L}(\rho) = \frac{\gamma}{2}(\sigma_{\mathbf{n}}\rho\sigma_{\mathbf{n}} - \rho)$?

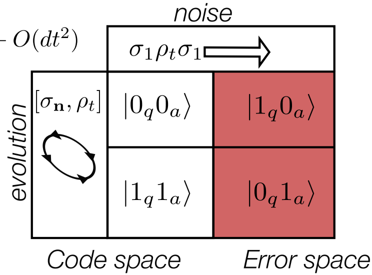
Result 2: Rank-one Pauli noise can be corrected without cancelling the evolution completely, but slowing it down:

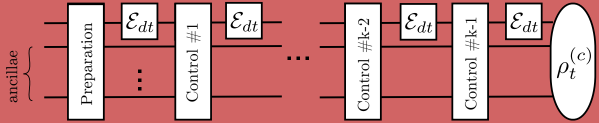
$$\omega \to \sqrt{1-(\mathbf{n}\cdot\mathbf{z})^2}\,\omega$$

$$\rho_{t+dt} = \rho_t(1-\tfrac{\gamma}{2}dt) - idt\,\tfrac{\omega}{2}[\sigma_{\mathbf{n}}, \rho_t] + dt\tfrac{\gamma}{2}\sigma_1\rho_t\sigma_1 + O(dt^2)$$

$$\Pi_C\rho_{t+dt}\Pi_C + \sigma_1\Pi_E\rho_{t+dt}\Pi_E\sigma_1$$

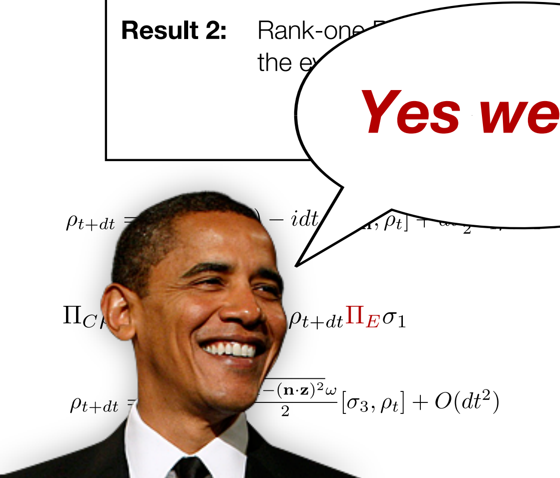
$$\rho_{t+dt} = \rho_t - idt\,\frac{\sqrt{1-(\mathbf{n}\cdot\mathbf{z})^2}\,\omega}{2}[\sigma_3, \rho_t] + O(dt^2)$$





What about rank-one Pauli noise $\mathcal{L}(\rho) = \frac{\gamma}{2}(\sigma_{\mathbf{n}}\rho\sigma_{\mathbf{n}} - \rho)$?

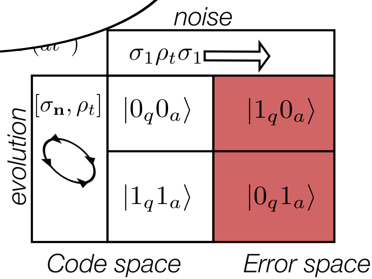
Result 2: Rank-one ... d without cancelling the e... wn:



$\rho_{t+dt} = \ldots) - idt \ldots, \rho_t] + \ldots (dt^2)$

$\Pi_C \ldots \rho_{t+dt} \Pi_E \sigma_1$

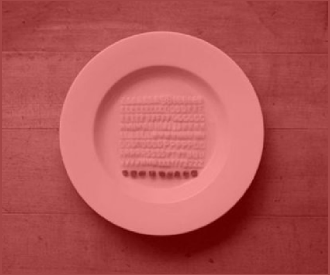
$\rho_{t+dt} = \ldots \frac{-(\mathbf{n}\cdot\mathbf{z})^2\omega}{2}[\sigma_3, \rho_t] + O(dt^2)$



Scaling of the QFI

Rank-one Pauli noise

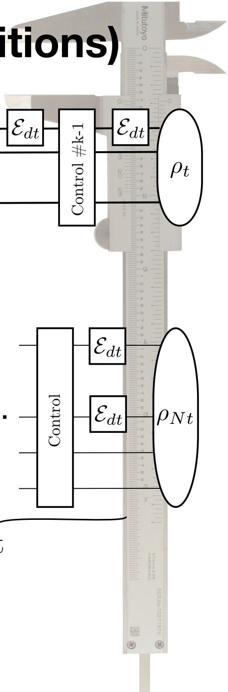
$$\mathcal{F}_{\mathbf{n}} = t^2(1 - (\mathbf{n} \cdot \mathbf{z})^2)$$



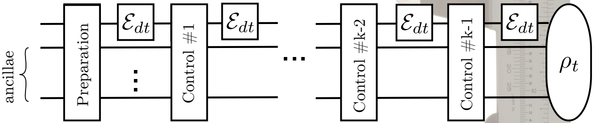
Any other noise

$$\mathcal{F}_{\mathcal{L}} \leq 4\,\alpha_{\mathcal{L}} t$$

Attainable precision (free repetitions)



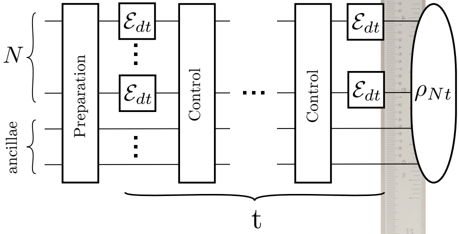
1.Time-particles



$$\nu\delta\omega^2 \geq \frac{1}{(1-(\mathbf{n}\cdot\mathbf{z})^2)t^2}$$

$$\nu\delta\omega^2 \geq \frac{1}{4\alpha_{\mathcal{L}}t}$$

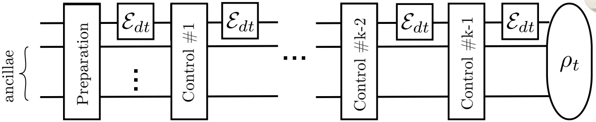
2.Frequency estimation

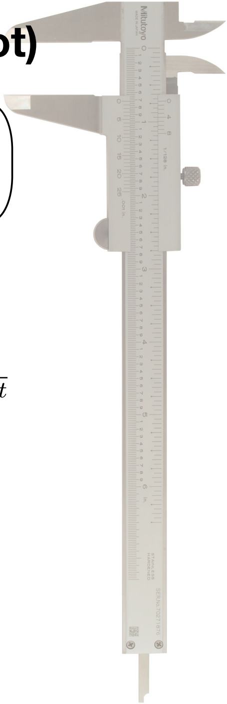


$$T\delta\omega^2 \geq \frac{1}{4\alpha_{\mathcal{L}}N}$$

$$T\delta\omega^2 \geq \frac{1}{(1-(\mathbf{n}\cdot\mathbf{z})^2)N^2t}$$ with $T := \nu t \gg t$

Attainable precision (single-shot)





Bayesian Cramer-Rao

$$\langle \delta\omega^2 \rangle \geq \frac{1}{4\alpha_{\mathcal{L}} t + \mathcal{F}(p(\omega))}$$

Ziv-Zakai (for QFI)

$$\langle \delta\omega^2 \rangle \geq \frac{1}{12\alpha_{\mathcal{L}} t}$$

Explicit strategy for interval prior

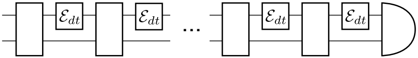
$$\langle \delta\omega^2 \rangle \leq \frac{165}{(1-(\mathbf{n}\cdot\mathbf{z})^2)t^2}$$

3.Qunatum control for metrology with limited resources

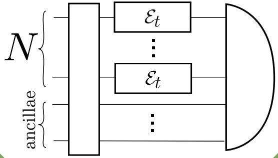
FFQC with limited ressource

Lindbladian: X-Y noise $\mathcal{L}(\rho) = \frac{\gamma}{2}(p\,\sigma_1\rho\sigma_1 + (1-p)\sigma_2\rho\sigma_2 - \rho)$

Sequential strategy with 2 qubits and FFQC



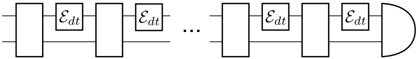
Parallel strategy with N qubits and ancillae



FFQC with limited ressource

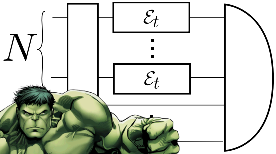
Lindbladian: X-Y noise $\mathcal{L}(\rho) = \frac{\gamma}{2}(p\,\sigma_1\rho\sigma_1 + (1-p)\sigma_2\rho\sigma_2 - \rho)$

Sequential strategy with
2 qubits and FFQC



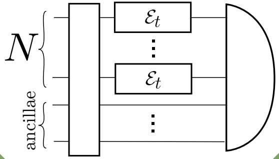
VS

Parallel strategy with
N qubits and ancillae



1.Solve the master equation

Parallel strategy with N qubits and ancillae

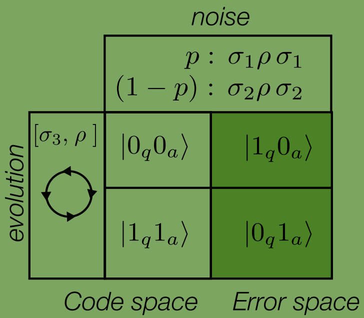


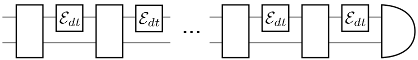
2. Optimize the bound numerically for a fixed *t*

$$\mathcal{F}(t) \leq \hat{\mathcal{F}}_{N,t} = 4N\,||\alpha(t)|| + 4N(N-1)\,||\beta(t)||^2$$

3. Optimize over *t*

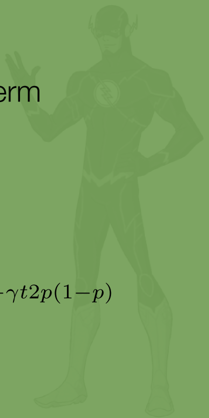
$$\hat{f}_N = \max_t \frac{\hat{\mathcal{F}}_{N,t}}{tN}$$





$$\mathcal{L}(\rho) = \frac{\gamma}{2}(p\,\sigma_1 \rho \sigma_1 + (1-p)\sigma_2 \rho \sigma_2 - \rho)$$

Sequential strategy with
2 qubits and FFQC



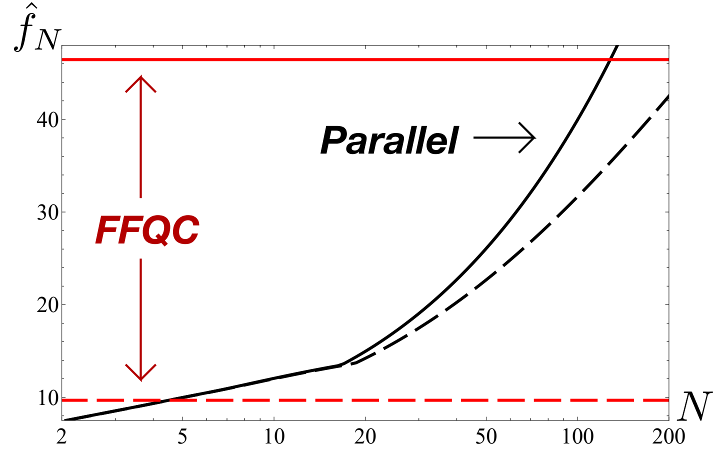
Detect the noise & Correct the most probable term

$$\rho_t = \bigoplus_m p(m;t) U_t \rho_m U_t^\dagger$$

Measure out the qubit at the optimal time

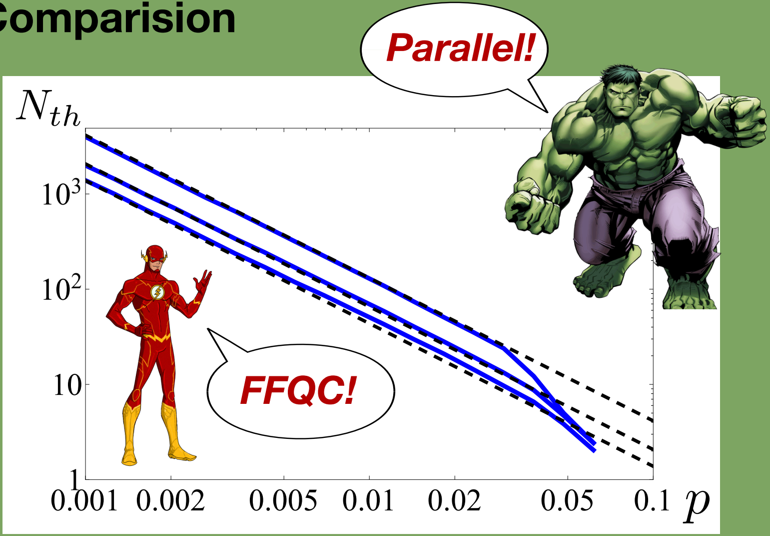
$$\mathcal{F}_t = t^2 \sum_m p(m;t) \mathcal{F}(\rho_m) = t^2 e^{-\gamma t 2p(1-p)}$$

Comparision



For p= 0.01(solid) and p= 0.05(dashed)

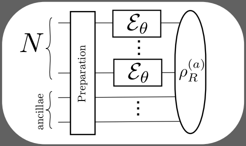
Comparision



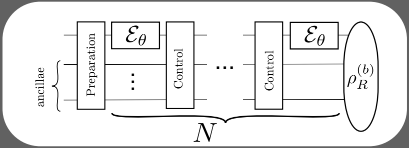
For $\omega = 2$ and $\gamma = \{0.1, 0.2, 0.3\}$

4.Role of intermediate control for phase estimation

Phase estimation



parallel

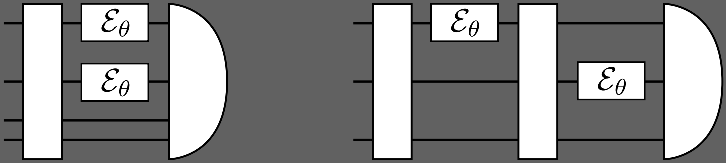


sequential

Is there an improvement?

Simple example

Two qubits and two ancilae



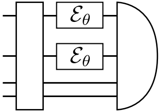
X-Y noise channel

$$\mathbf{K} = (\sqrt{p}\,U_\theta, \sqrt{\tfrac{1-p}{2}}\sigma_1, \sqrt{\tfrac{1-p}{2}}\sigma_2) \simeq (\sqrt{p}\,U_\theta, \sqrt{1-p}|1\rangle\langle 0|, \sqrt{1-p}|0\rangle\langle 1|)$$

Optimal encoding

$$|0\rangle \to |0_q 0_a\rangle$$

$$|1\rangle \to |1_q 1_a\rangle$$

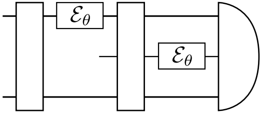


Find the optimal entangled state

$$|\psi\rangle_{opt} = \sqrt{\tfrac{x_{opt}}{2}}(|00\rangle_Q|00\rangle_S + |11\rangle_Q|11\rangle_S) + \sqrt{\tfrac{1-x_{opt}}{2}}(|01\rangle_Q|01\rangle_S + |10\rangle_Q|10\rangle_S)$$

The corresponding QFI

$$\mathcal{F} = \begin{cases} -\frac{(p-2)^2 p}{2(p-1)} & p \leq \frac{2}{3} \\ 4p^2 & otherwise. \end{cases}$$



Prepare the first qubit

$$|\psi\rangle_1 = \sqrt{\tfrac{1}{2}}(|0\rangle_{Q_1}|0\rangle_A + |1\rangle_{Q_1}|1\rangle_A)$$

Read out the error syndrome

E: Prepare the same state for the second qubit.

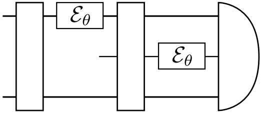
$$|\psi\rangle_2 = \sqrt{\tfrac{1}{2}}(|0\rangle_{Q_2}|0\rangle_A + |1\rangle_{Q_2}|1\rangle_A)$$

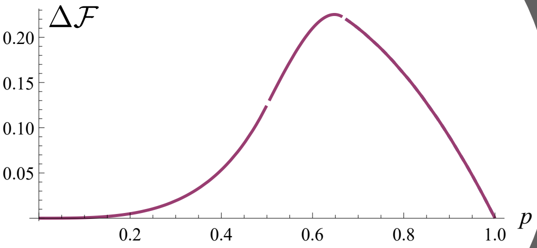
C: Entangle the second qubit with the first one.

$$\sqrt{\tfrac{1}{2}}\Big(e^{-i\frac{\theta}{2}}|0\rangle_{Q_1}(\sqrt{y}|00\rangle_{Q_2,A} + \sqrt{1-y}|11\rangle_{Q_2,A}) + e^{i\frac{\theta}{2}}|1\rangle_{Q_1}(\sqrt{y}|11\rangle_{Q_2,A} + \sqrt{1-y}|00\rangle_{Q_2,A})\Big)$$

Prepare t

$|\psi\rangle_1 =$





ngle the second qubit
n the first one.

$|\psi\rangle$

$$(\sqrt{y}|00\rangle_{Q_2,A} + \sqrt{1-y}|11\rangle_{Q_2,A})$$

$$|1\rangle_{Q_1}(\sqrt{y}|11\rangle_{Q_2,A} + \sqrt{1-y}|00\rangle_{Q_2,A})\Big)$$

1. Scaling of the QFI in presence of noise. Implications for precision.

2. Role of FFQC for metrology with limited resources.

3. Role of intermediate control for phase estimation.