Quantum Metrology Kills Rayleigh's Criterion *

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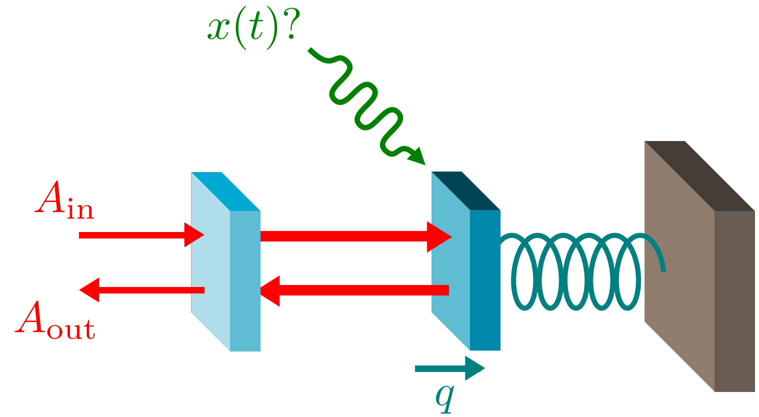
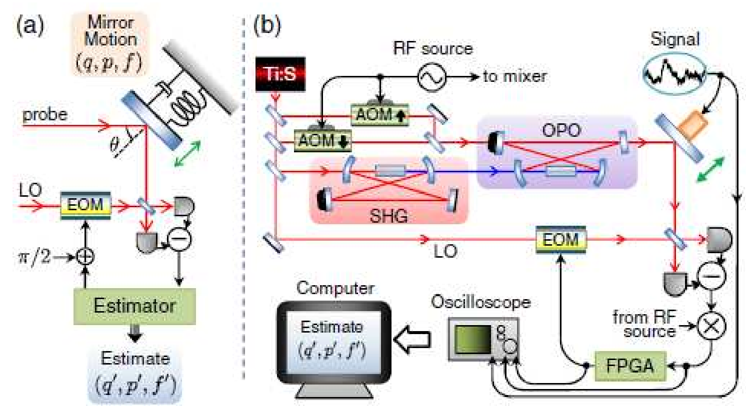
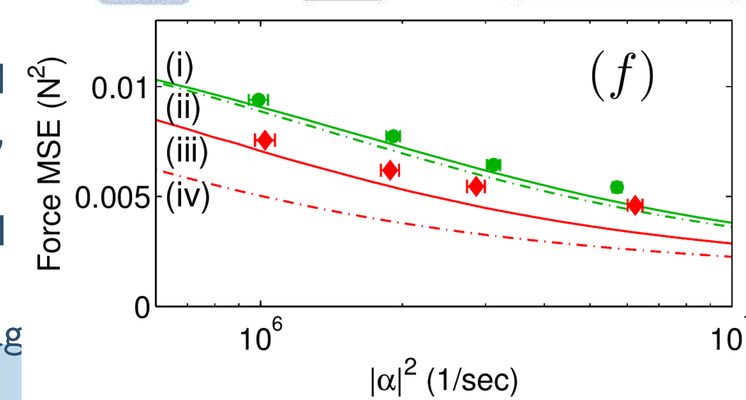
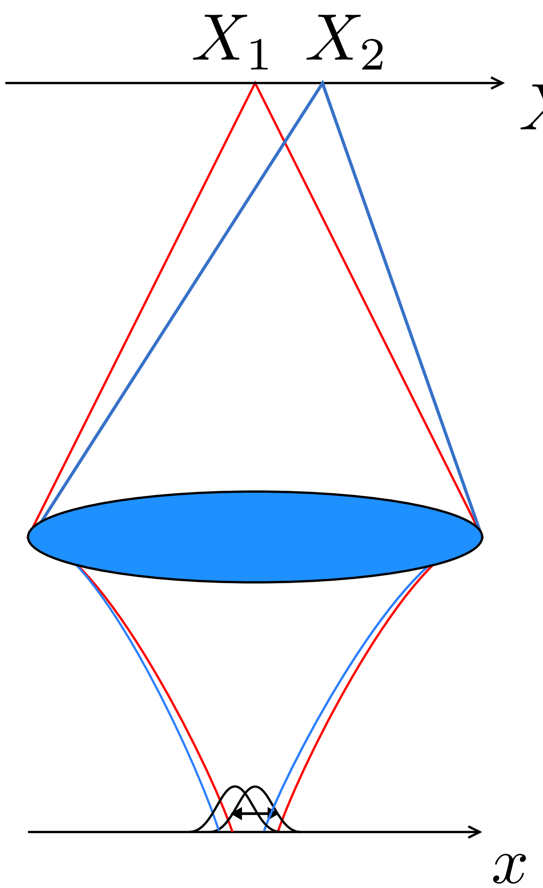
Quantum Metrology for Dynamical Systems, e.g., Gravitational-Wave Detectors, Atomic Clocks

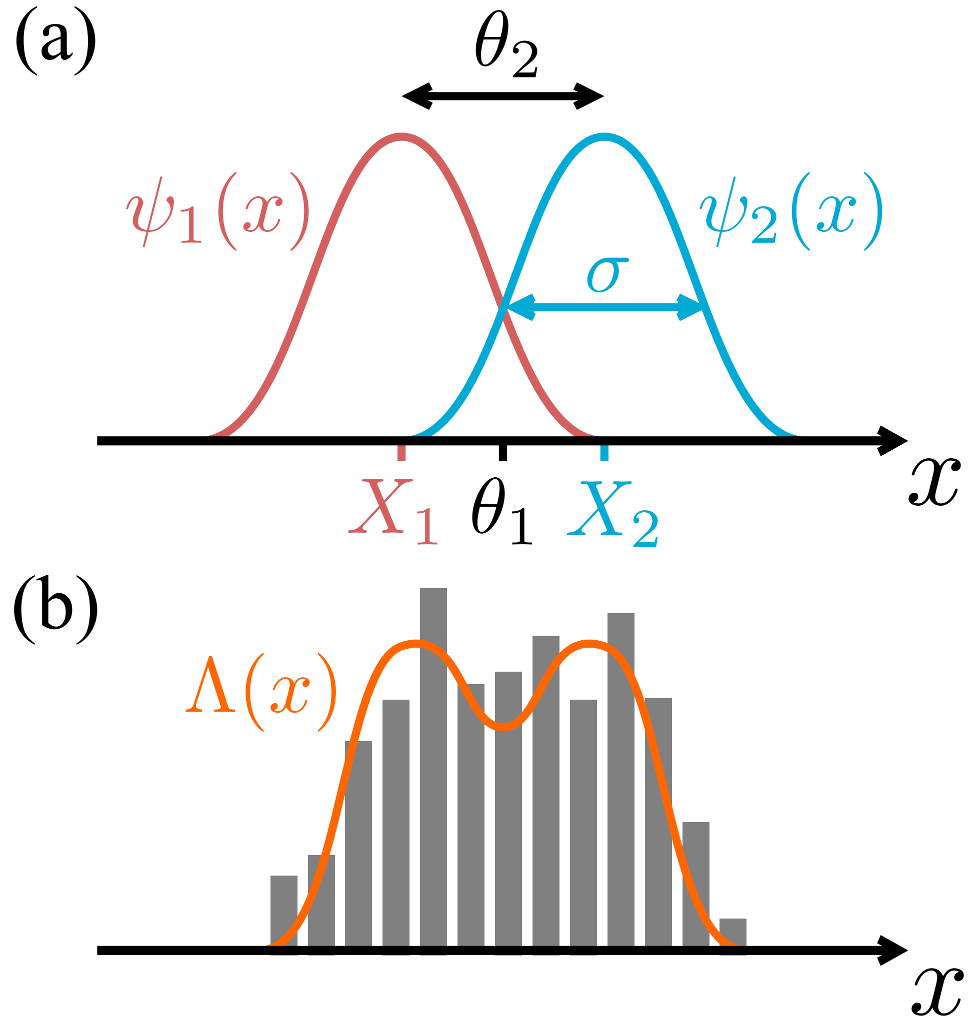
- Tsang, Wiseman, and Caves, “Fundamental Quantum Limit to Waveform Estimation,” PRL **106**, 090401 (2011).
  - Bayesian QCRB for stochastic process (**infinite** number of parameters), **non-commuting** Heisenberg-picture generators, **sequential/continuous** measurements
- Tsang and Nair, ‘Fundamental quantum limits to waveform detection,” PRA **86**, 042115 (2012).
  - Helstrom bounds on waveform detection/false-alarm errors
- Tsang, “Quantum metrology with open dynamical systems,” NJP **15**, 073005 (2013).
- Iwasawa *et al.*, “Quantum-limited mirror-motion estimation,” PRL **111**, 163602 (2013).
- Berry *et al.*, “The Quantum Bell-Ziv-Zakai Bounds and Heisenberg Limits for Waveform Estimation,” PRX **5**, 031018 (2015).
- Ng *et al.*, “Spectrum analysis with quantum dynamical systems,” preprint available upon request.
- More on `https://sites.google.com/site/mankeitsang`

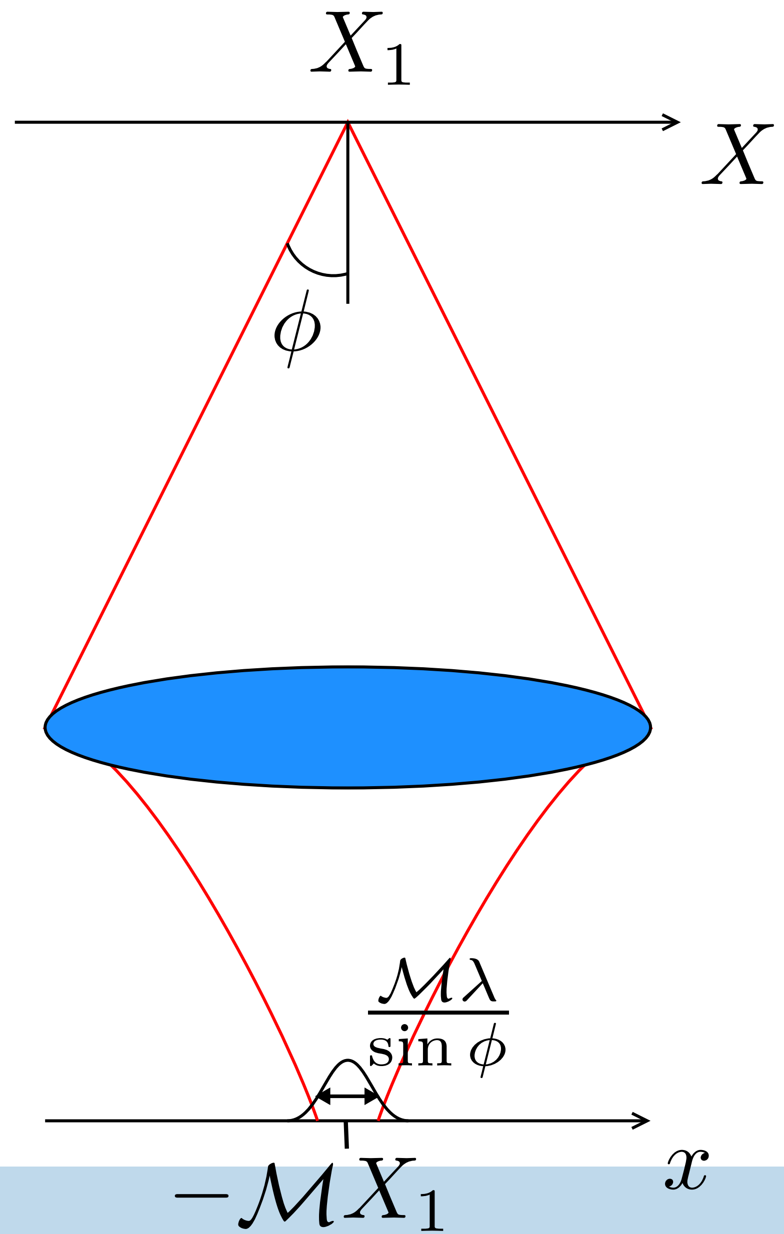




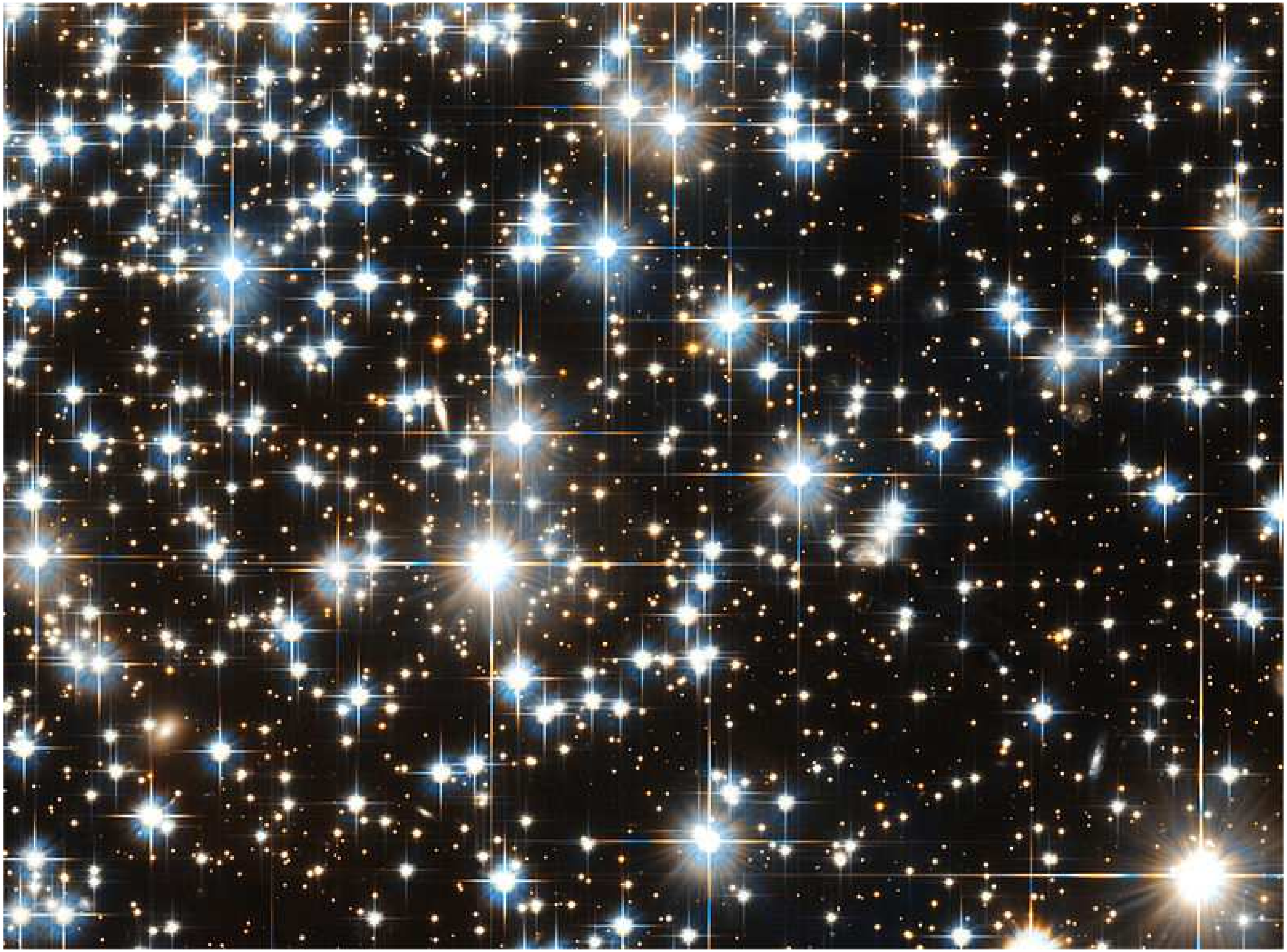



X_1 X_2
X
x


(a)
θ_2
$\psi_1(x)$
$\psi_2(x)$
σ
X_1 θ_1 X_2
x
(b)
$\Lambda(x)$
x


X_1
X
ϕ
$\frac{\mathcal{M}\lambda}{\sin\phi}$
$-\mathcal{M}X_1$
x

Point-Spread Function of Hubble Space Telescope

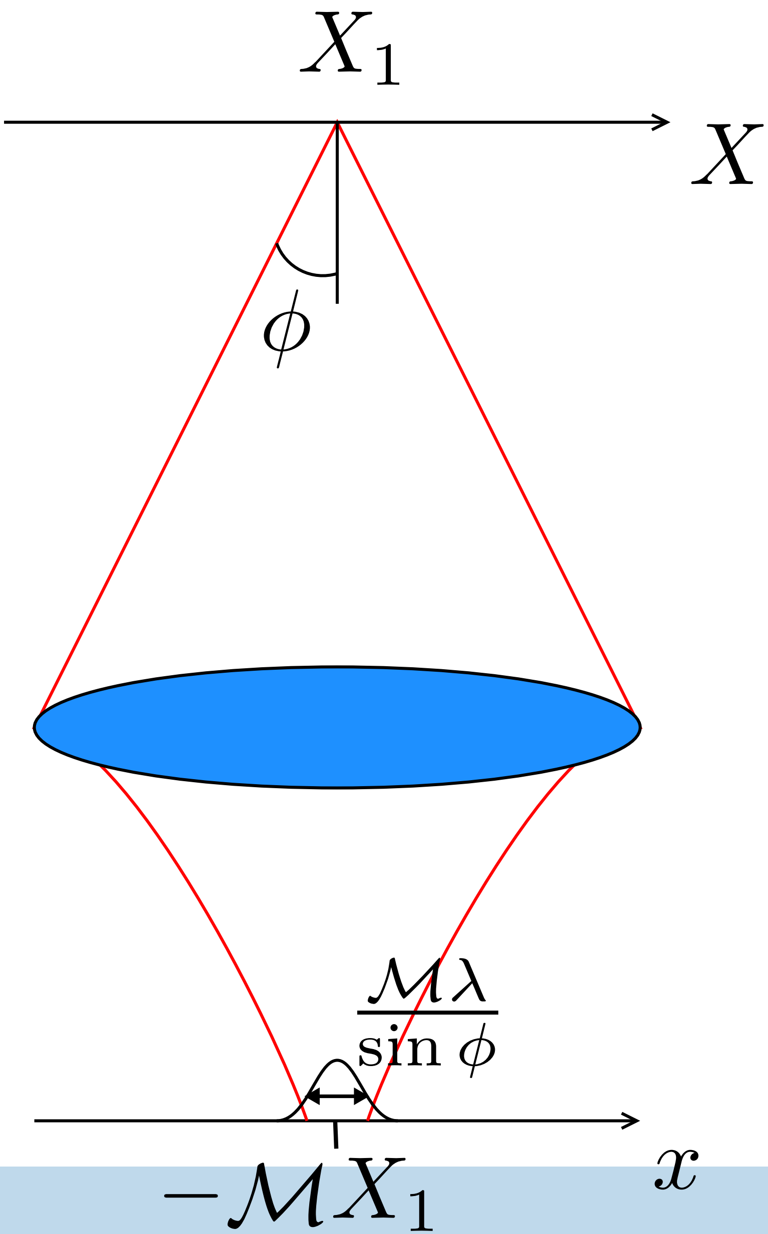


Inferring Position of One Point Source

- Given N detected photons on camera, **mean-square error** of estimating X_1:

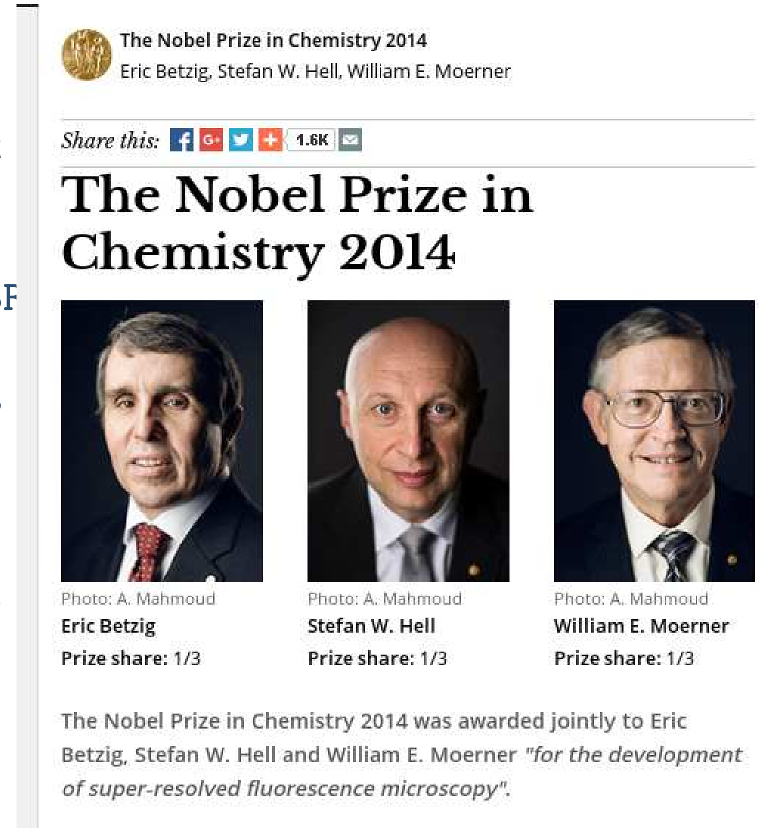
$$\Delta X_1^2 = \frac{\sigma^2}{N}, \qquad \sigma \sim \frac{\lambda}{\sin\phi}. \tag{1}$$

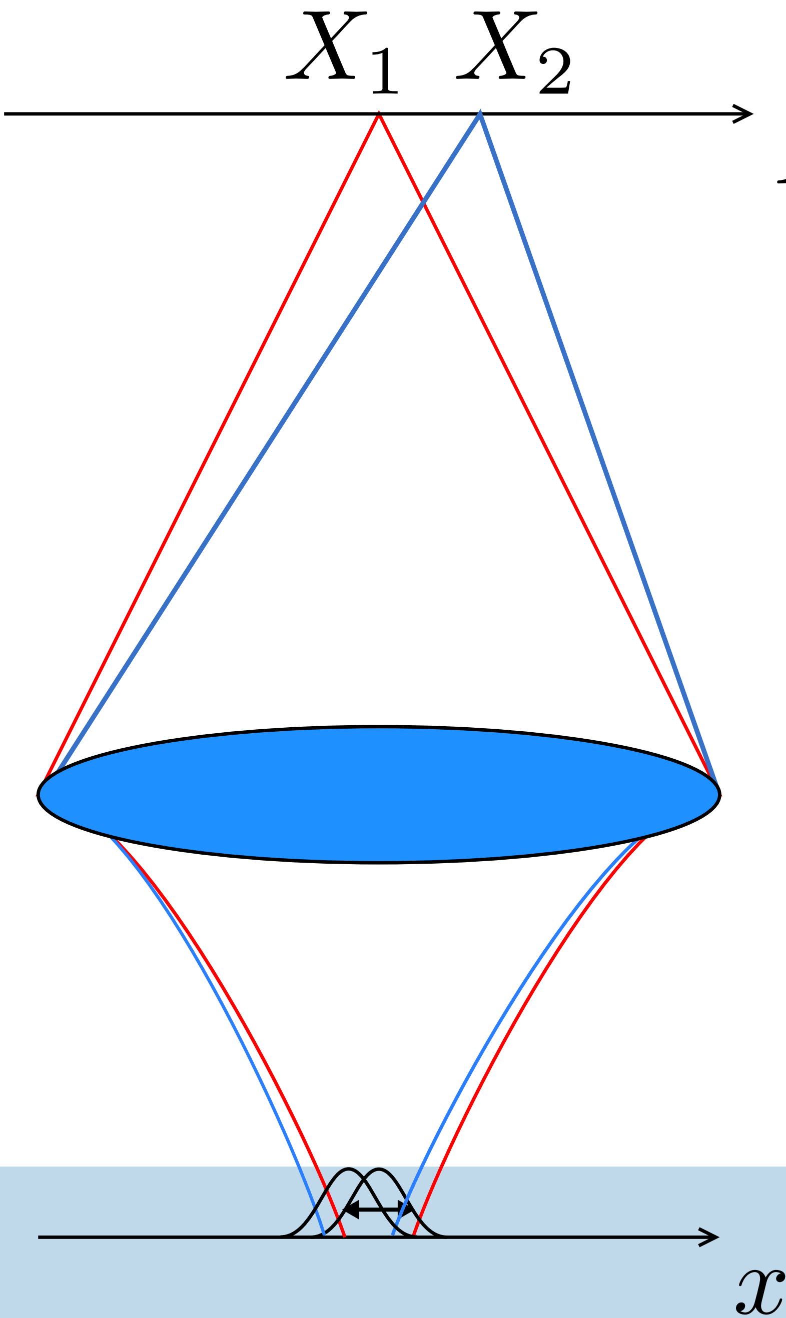
- Assume classical source (e.g., thermal source, laser source).
- Derive from signal-to-noise ratio, classical/quantum Cramér-Rao bounds, etc.
- Helstrom

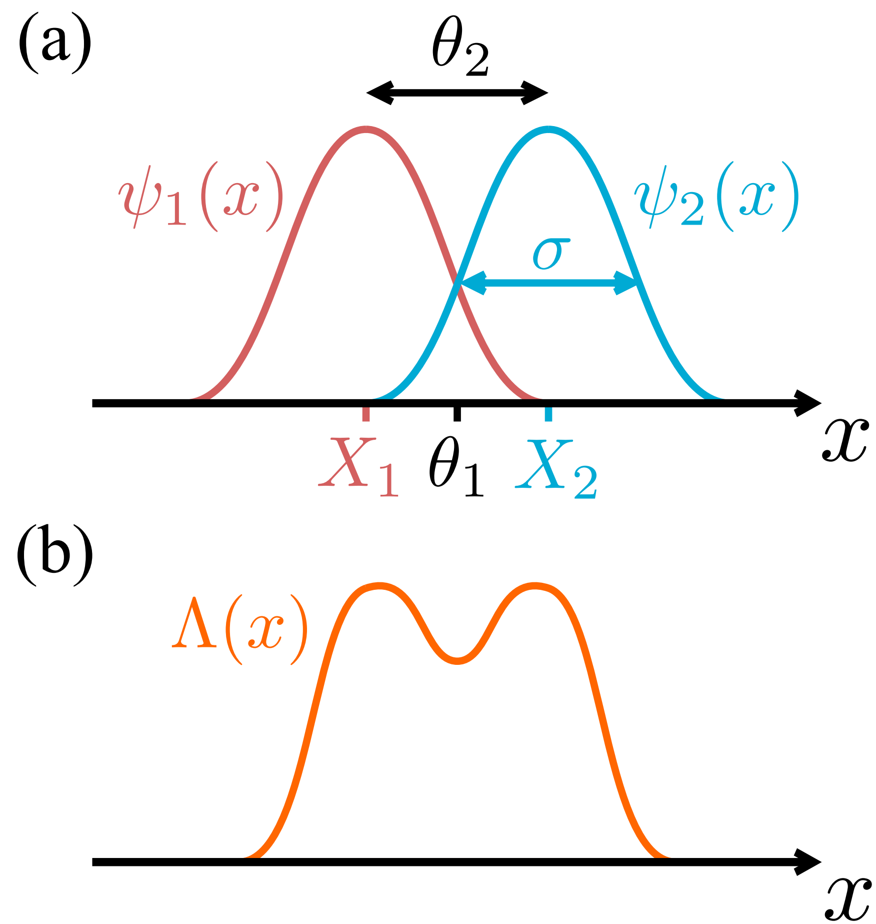


Superresolution Microscopy

- PALM, STED, STORM, etc.: Make fluorescent particles radiate in **isolation**. Estimate their positions accurately by locating the **centroids**.
- `https://www.youtube.com/watch?v=2R2l19SF` (25:45)
- require special fluorescent particles (blinking/bleaching/stimulated-emission-depleted, doesn't work for stars), slow
- e.g., Betzig, Optics Letters **20**, 237 (1995).
- For a review, see Moerner, PNAS **104**, 12596 (2007).







- Rayleigh's criterion (1879): requires $\theta_2 \gtrsim \sigma$ (heuristic)

Centroid and Separation Estimation

- Bettens *et al.*, Ultramicroscopy **77**, 37 (1999); Ram, Ward, Ober, PNAS **103**, 4457 (2006)
- Assume incoherent sources, Poissonian statistics for CCD
- classical Cramér-Rao bound for **centroid**:

$$\Delta\theta_1^2 \sim \frac{\sigma^2}{N}. \tag{2}$$

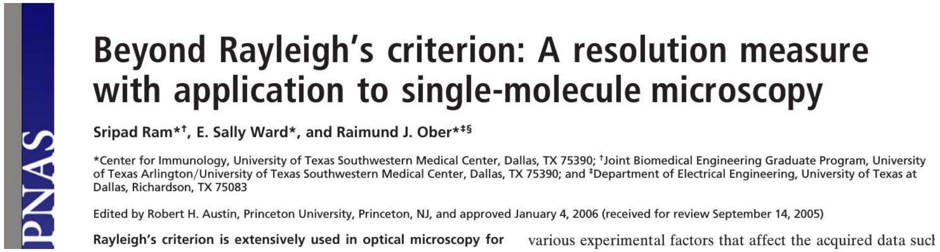
- CRB for **separation estimation**: two regimes
  - $\theta_2 \gg \sigma$:

$$\Delta\theta_2^2 \approx \frac{4\sigma^2}{N}, \tag{3}$$

  - $\theta_2 \ll \sigma$:

$$\boxed{\Delta\theta_2^2 \to \frac{4\sigma^2}{N} \times \infty} \tag{4}$$

    Bound blows up when the two spots overlap significantly.
  - **Curse of Rayleigh's criterion**
  - Localization microscopy: must avoid overlapping spots.


PNAS

Beyond Rayleigh's criterion: A resolution measure with application to single-molecule microscopy

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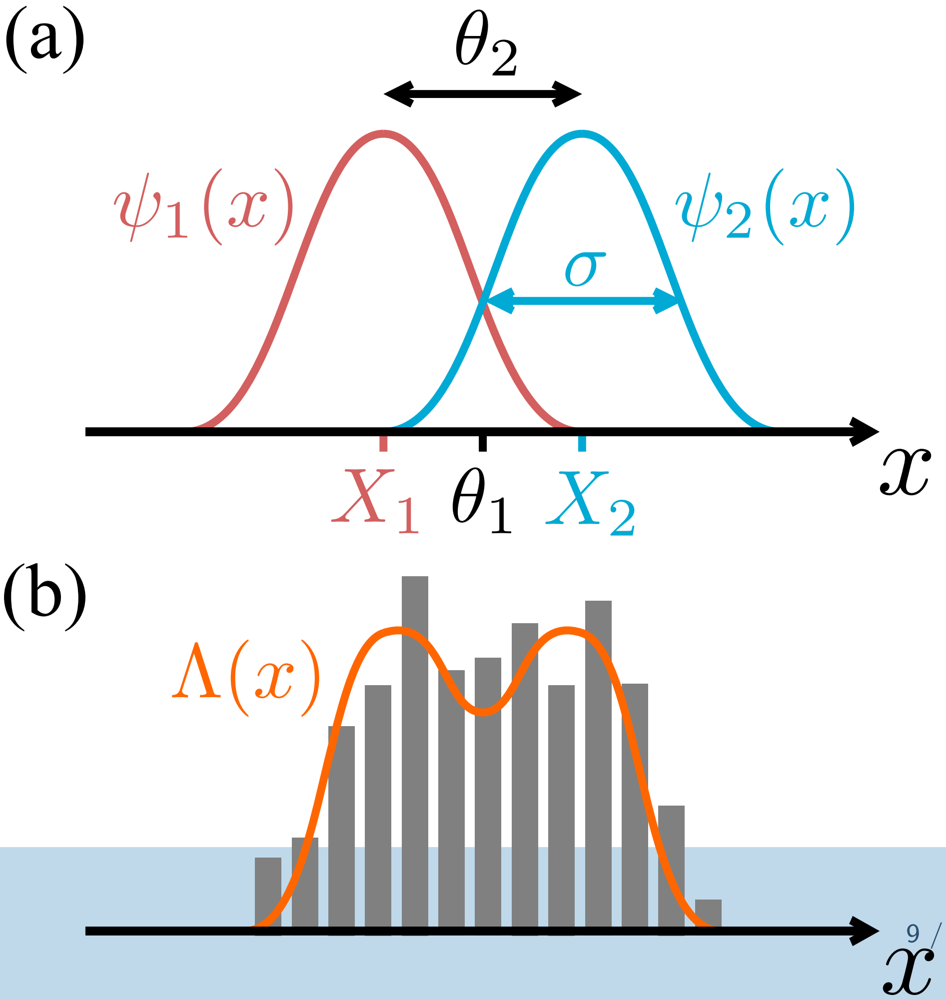
Rayleigh's criterion is extensively used in optical microscopy for various experimental factors that affect the acquired data sucl

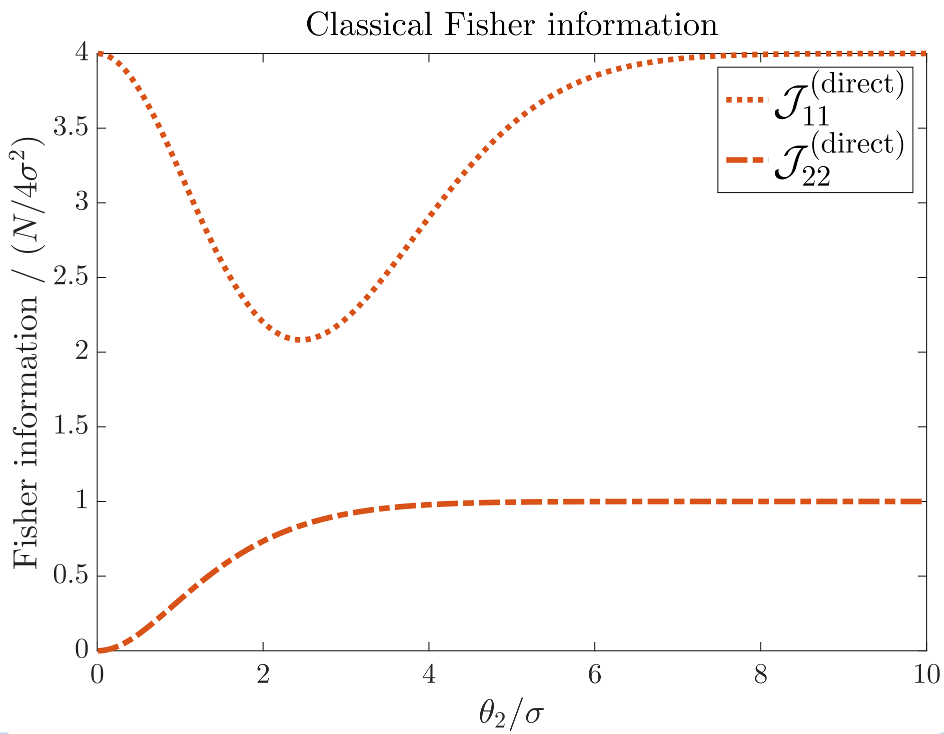


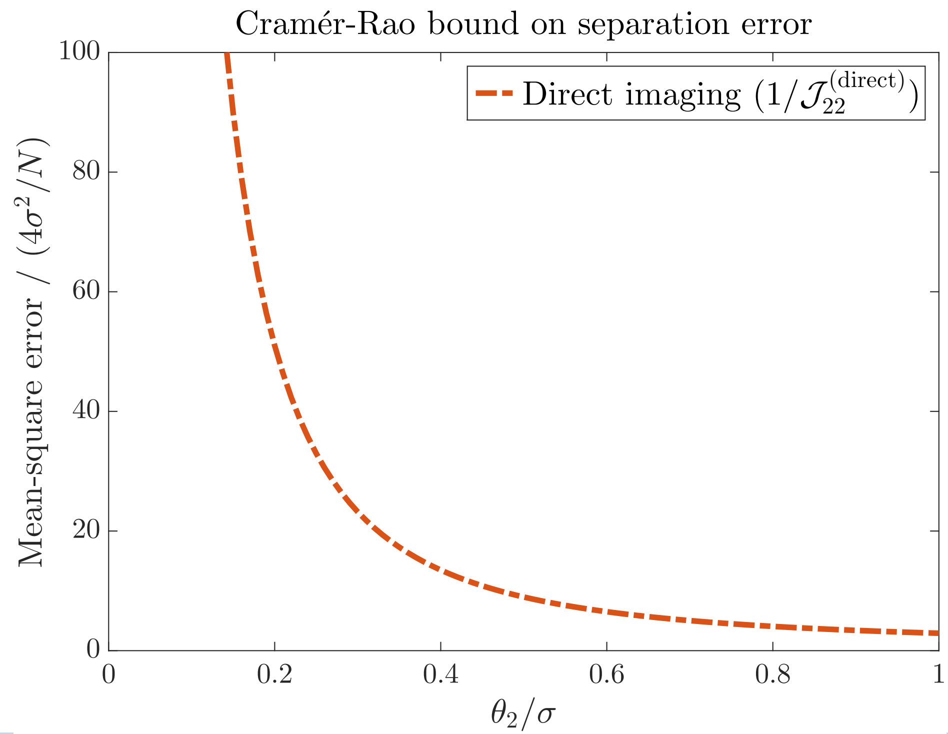
Fisher Information and Cramér-Rao Bound

- Cramér-Rao bounds:

$$\Delta\theta_1^2 \geq \frac{1}{\mathcal{J}_{11}^{(\mathrm{direct})}} \qquad \Delta\theta_2^2 \geq \frac{1}{\mathcal{J}_{22}^{(\mathrm{direct})}} \tag{5}$$

  $\mathcal{J}^{(\mathrm{direct})}$ is Fisher information for CCD (direct imaging via spatially resolved photon counting).
- Assume Gaussian PSF, similar behavior for other PSF
- **Rayleigh's curse**: $\Delta\theta_2$ blows up when Rayleigh's criterion is violated.





Quantum Cramér-Rao Bounds

- CCD is just one measurement method. **Quantum mechanics allows infinite possibilities.**
- Helstrom: For any measurement (POVM) of an optical state $\rho^{\otimes M}$ (M here is number of copies, not magnification)

$$\Sigma \geq \mathcal{J}^{-1} \boxed{\geq \mathcal{K}^{-1},} \tag{6}$$

$$\mathcal{K}_{\mu\nu} = M \,\mathrm{Re}\left(\mathrm{tr}\, \mathcal{L}_\mu \mathcal{L}_\nu \rho\right), \tag{7}$$

$$\frac{\partial \rho}{\partial \theta_\mu} = \frac{1}{2}\left(\mathcal{L}_\mu \rho + \rho \mathcal{L}_\mu\right). \tag{8}$$

- Coherent sources: Tsang, Optica **2**, 646 (2015).
- Mixed states:

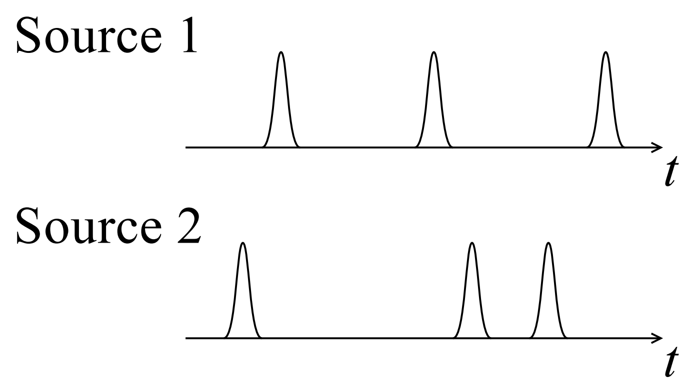
$$\rho = \sum_n D_n \left|e_n\right\rangle \left\langle e_n\right|, \tag{9}$$

$$\mathcal{L}_\mu = 2 \sum_{n,m;D_n+D_m\neq 0} \frac{\left\langle e_n\right| \frac{\partial \rho}{\partial \theta_\mu} \left|e_m\right\rangle}{D_n + D_m} \left|e_n\right\rangle \left\langle e_m\right|. \tag{10}$$

- Helstrom didn't study this particular two-source quantum parameter estimation problem.

Quantum Optics for Thermal Sources

- Mandel and Wolf, *Optical Coherence and Quantum Optics*; Goodman, *Statistical Optics*
- Consider a thermal source, e.g., stars, fluorescent particles.
- Coherence time ~ 10 fs. Within each coherence time interval, **average photon number** $\epsilon \ll 1$ **at optical frequencies** (visible, UV, X-ray, etc.).



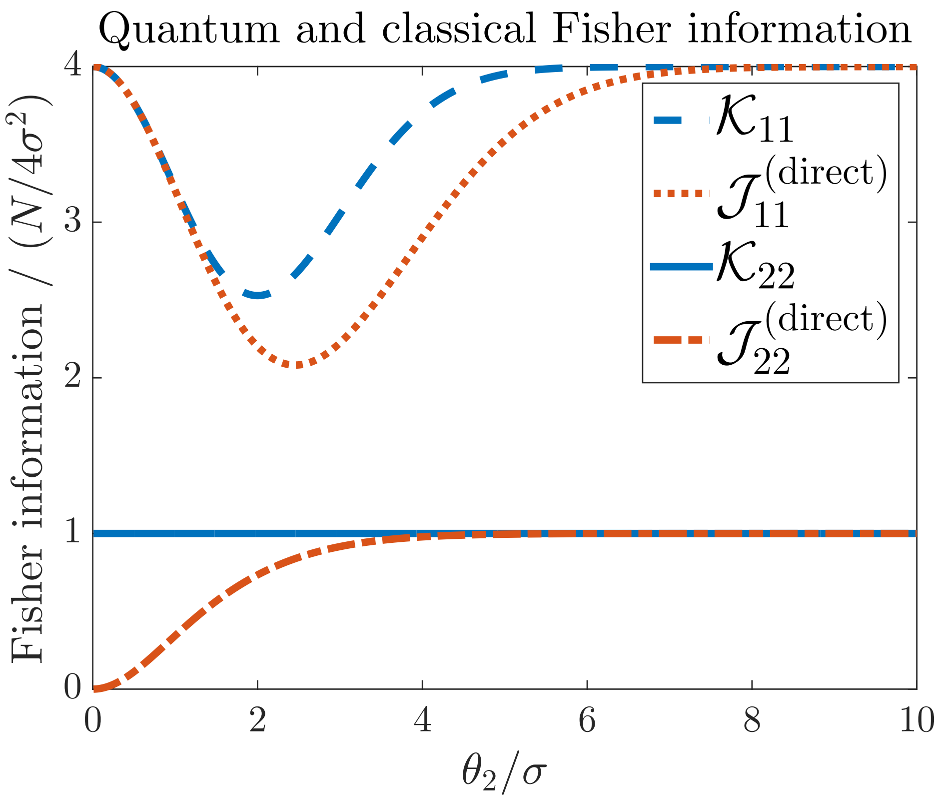
- Quantum state at image plane:

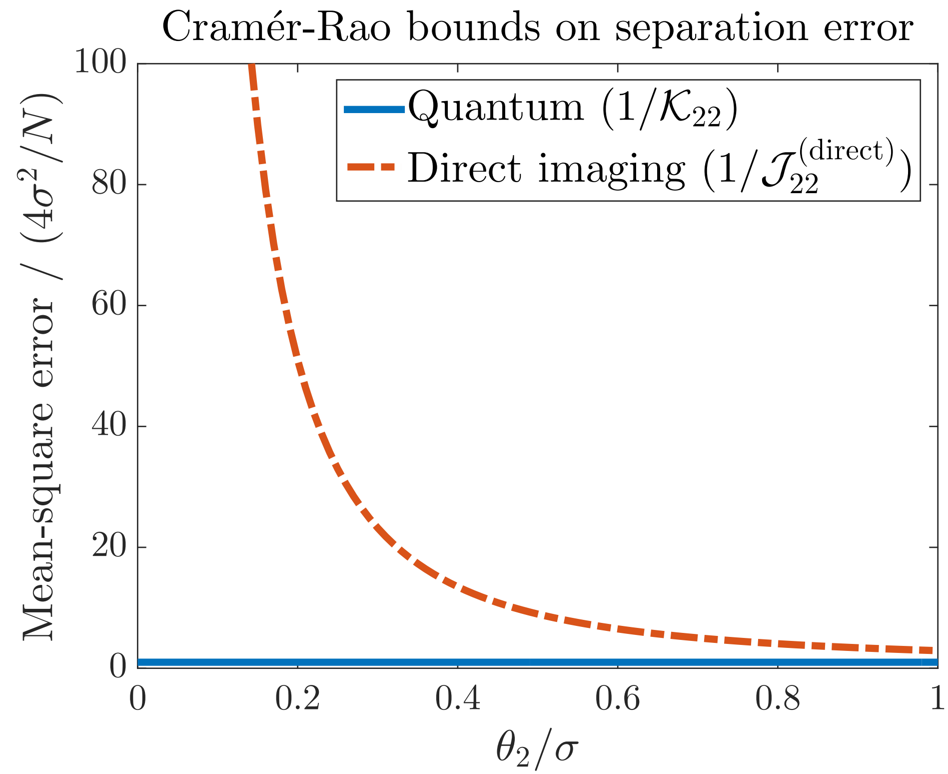
$$\boxed{\rho = (1-\epsilon)\left|\text{vac}\right\rangle\left\langle\text{vac}\right| + \frac{\epsilon}{2}\left(\left|\psi_1\right\rangle\left\langle\psi_1\right| + \left|\psi_2\right\rangle\left\langle\psi_2\right|\right) + O(\epsilon^2)} \qquad \boxed{\left\langle\psi_1|\psi_2\right\rangle \neq 0,} \tag{11}$$

$$\left|\psi_1\right\rangle = \int_{-\infty}^{\infty} dx \psi(x - X_1)\left|x\right\rangle, \quad \left|\psi_2\right\rangle = \int_{-\infty}^{\infty} dx \psi(x - X_2)\left|x\right\rangle, \quad \left|x\right\rangle = a^\dagger(x)\left|\text{vac}\right\rangle. \tag{12}$$

- derive from zero-mean Gaussian P function
- Multiphoton coincidence: **rare**, little information as $\epsilon \ll 1$ (homeopathy)
- Similar model for stellar interferometry in Gottesman, Jennewein, Croke, PRL **109**, 070503 (2012); Tsang, PRL **107**, 270402 (2011).

Classical and Quantum Cramér-Rao Bounds





- For any $\psi(x)$ with constant phase,

$$\boxed{\Delta\theta_2^2 \geq \frac{1}{\mathcal{K}_{22}} = \frac{1}{N\Delta k^2}.} \tag{13}$$

  (For Gaussian ψ, $\sigma = 1/(2\Delta k)$)
- Ranjith Nair (unpublished): $\mathcal{K}_{22}$ for arbitrary ϵ gives the same result to the first order of ϵ
- Hayashi *ed.*, *Asymptotic Theory of Quantum Statistical Inference*; Fujiwara JPA **39**, 12489 (2006): there exists a POVM such that $\Delta\theta_\mu^2 \to 1/\mathcal{K}_{\mu\mu}$, $M \to \infty$.

Hermite-Gaussian basis

- project the photon in **Hermite-Gaussian** basis:

$$E_1(q) = |\phi_q\rangle \langle\phi_q| \,, \tag{14}$$

$$|\phi_q\rangle = \int_{-\infty}^{\infty} dx \phi_q(x) |x\rangle \,, \tag{15}$$

$$\phi_q(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} H_q\left(\frac{x}{\sqrt{2}\sigma}\right) \exp\left(-\frac{x^2}{4\sigma^2}\right). \tag{16}$$

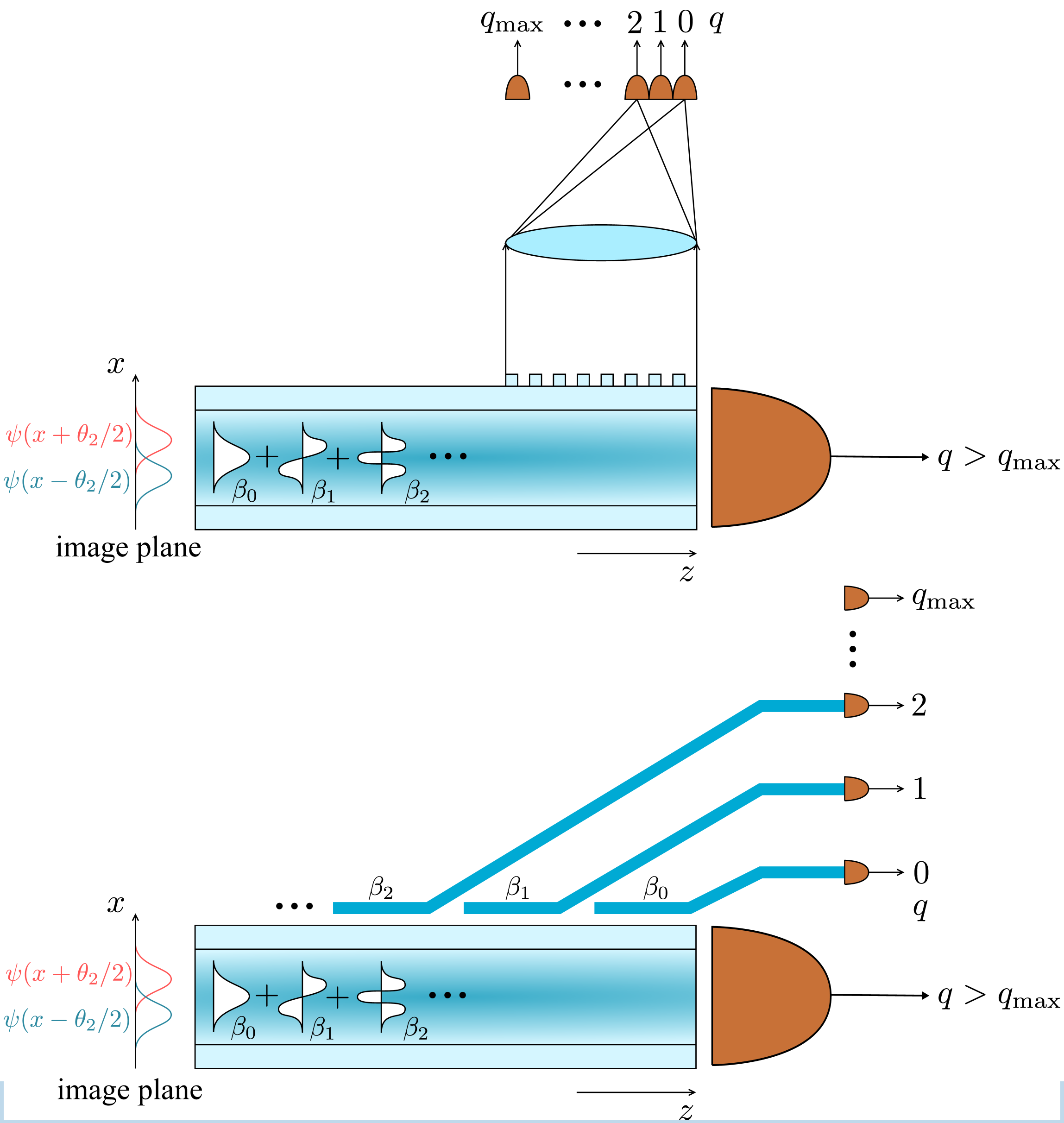
- Assume PSF $\psi(x)$ is Gaussian (common).

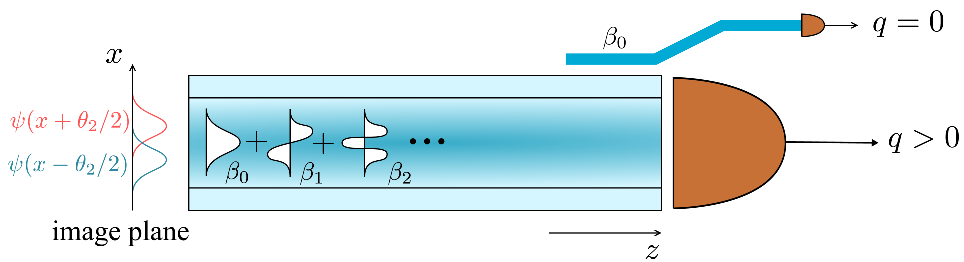
$$|\langle\phi_q|\psi_1\rangle|^2 = |\langle\phi_q|\psi_2\rangle|^2 = \left|\int_{-\infty}^{\infty} dx \phi_q^*(x)\psi\left(x \pm \frac{\theta_2}{2}\right)\right|^2 = \exp(-Q)\frac{Q^q}{q!}, \quad Q \equiv \frac{\theta_2^2}{16\sigma^2}. \tag{17}$$

$$\boxed{\frac{1}{\mathcal{J}_{22}^{(\mathrm{HG})}} = \frac{1}{\mathcal{K}_{22}} = \frac{4\sigma^2}{N}.} \tag{18}$$

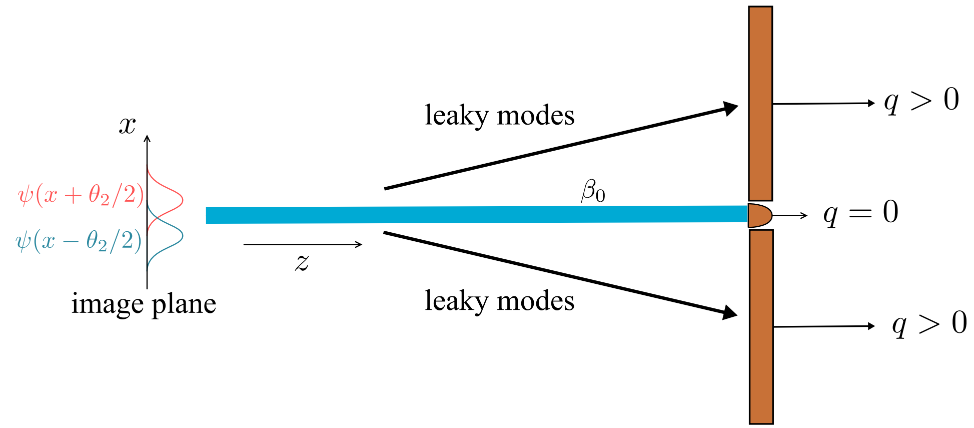
- **Maximum-likelihood estimator** can asymptotically saturate the classical bound.

Spatial-Mode Demultiplexing (SPADE)

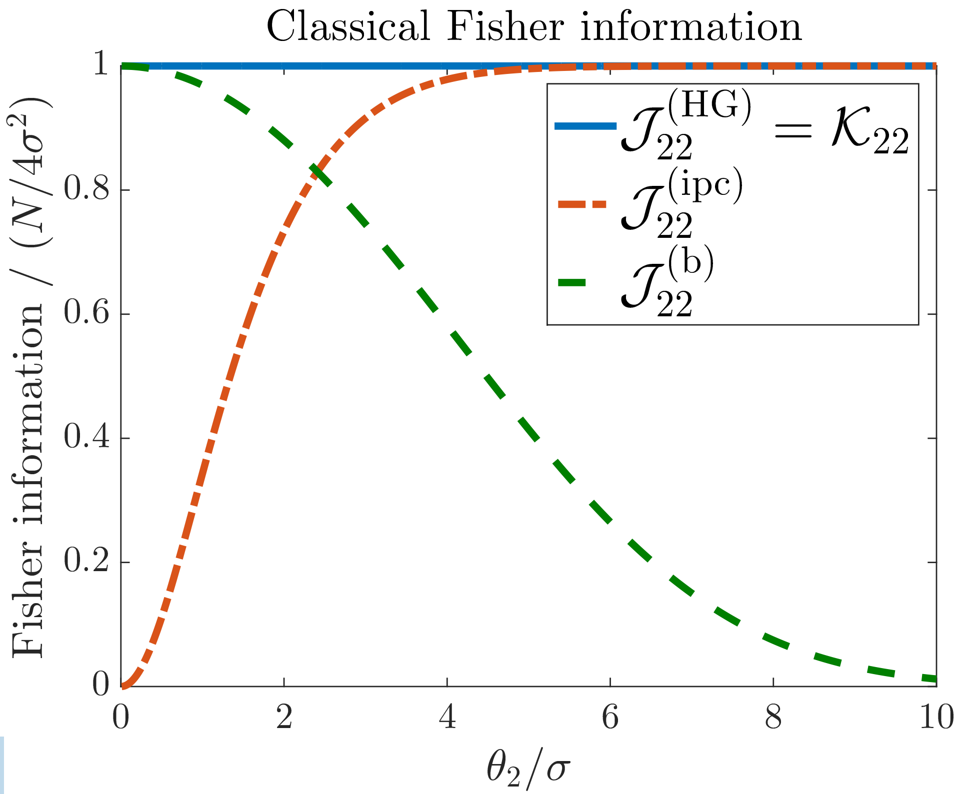




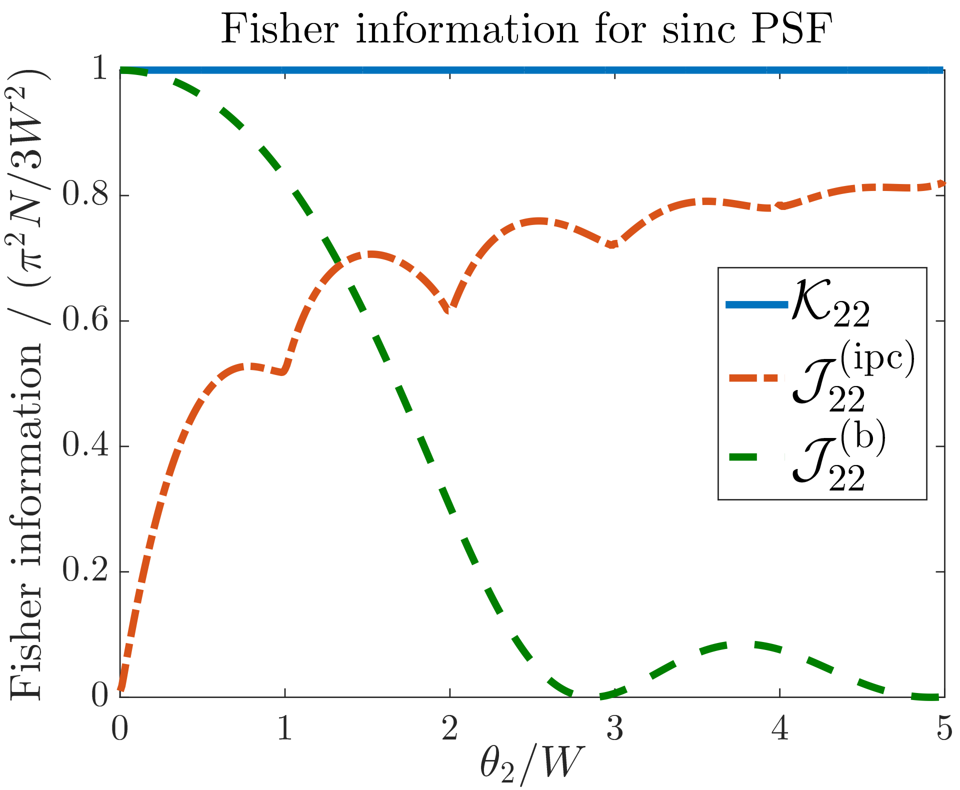
x
$\psi(x+\theta_2/2)$
$\psi(x-\theta_2/2)$
image plane
β_0 β_1 β_2 $\cdots$
β_0
$q=0$
$q>0$
z



x
$\psi(x+\theta_2/2)$
$\psi(x-\theta_2/2)$
image plane
leaky modes
β_0
z
leaky modes
$q>0$
$q=0$
$q>0$

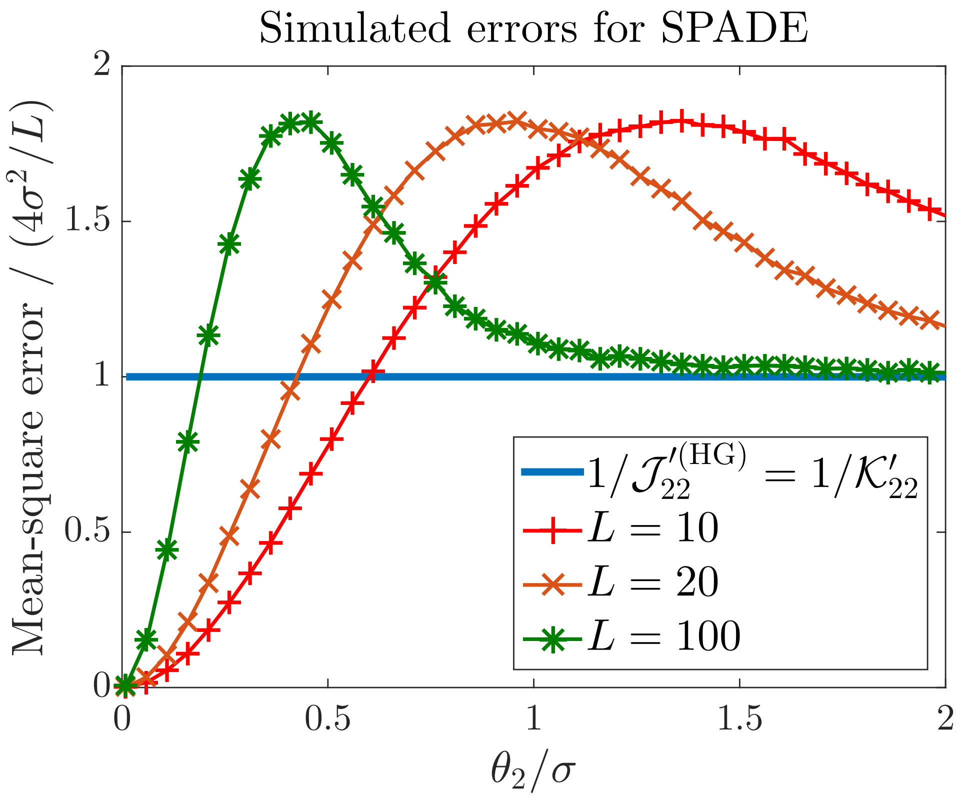


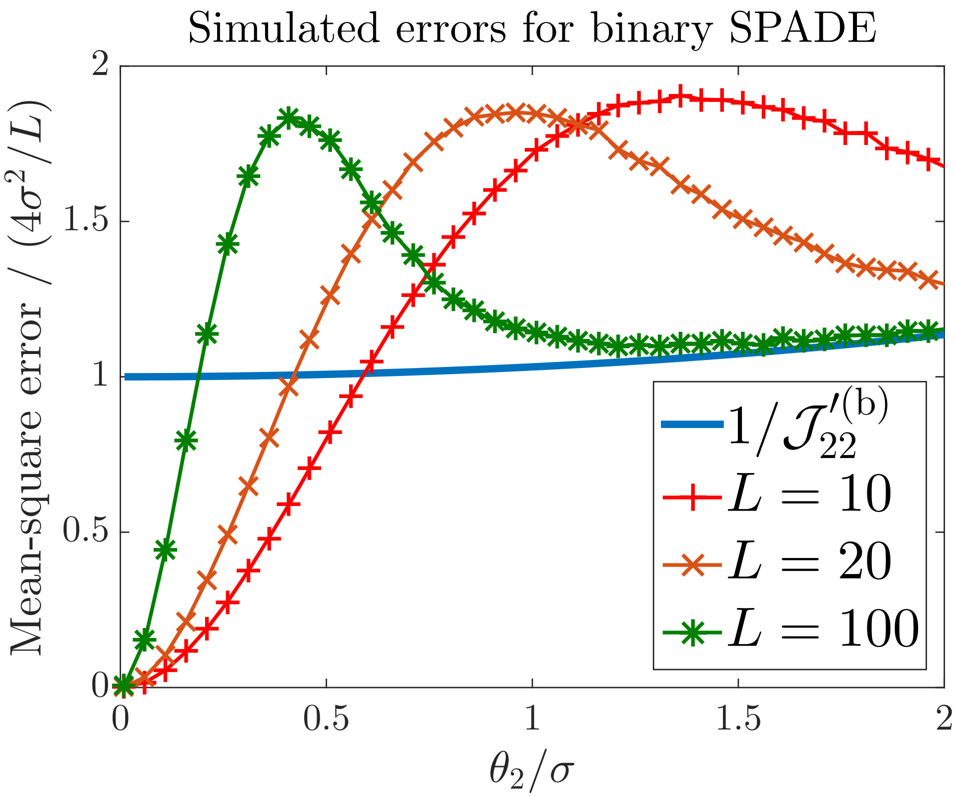
Classical Fisher information
Fisher information / $(N/4\sigma^2)$
θ_2/σ
$\mathcal{J}_{22}^{(\mathrm{HG})} = \mathcal{K}_{22}$
$\mathcal{J}_{22}^{(\mathrm{ipc})}$
$\mathcal{J}_{22}^{(\mathrm{b})}$



Fisher information for sinc PSF
Fisher information / $(\pi^2 N/3W^2)$
θ_2/W
$\mathcal{K}_{22}$
$\mathcal{J}_{22}^{(\mathrm{ipc})}$
$\mathcal{J}_{22}^{(\mathrm{b})}$

Numerical Performance of Maximum-Likelihood Estimators

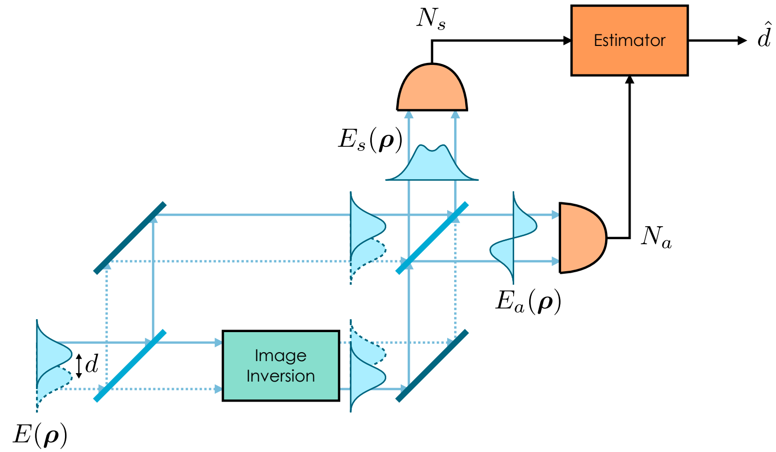




- $L =$ number of detected photons
- **biased** (in a good way), $< 2\times$CRB.

Extensions

- Tsang, Nair, and Lu "Quantum theory of superresolution for two incoherent optical point sources," arXiv:1511.00552 (submitted, stuck with a referee).
- Tsang, Nair, and Lu, "Semiclassical Theory of Superresolution for Two Incoherent Optical Point Sources," arXiv:1602.04655 (submitted to QCMC 2016): **semiclassical Poisson model** gives the same results, also works for **lasers**.
- Nair and Tsang, "Interferometric superlocalization for two incoherent optical point sources," Opt. Express **24**, 3684 (2016).



  - Super-Localization by Image-inVERsion interferometry (SLIVER)
  - 2D imaging, semiclassical theory, **arbitrary** ϵ.
  - covered by **Laser Focus World**, Feb 2016.

- Ang, Nair, Tsang (submitted to CLEO/QELS 2016)
  - QCRB for **2D** and $\epsilon \ll 1$
  - 2D generalizations of SPADE and SLIVER
- Nair, Lu, and Tsang (under preparation)
  - 1D and **arbitrary** ϵ
  - Fidelity for hypothesis testing (one source vs two sources), extending Helstrom
  - QCRB (validates our $\epsilon \ll 1$ approximation)

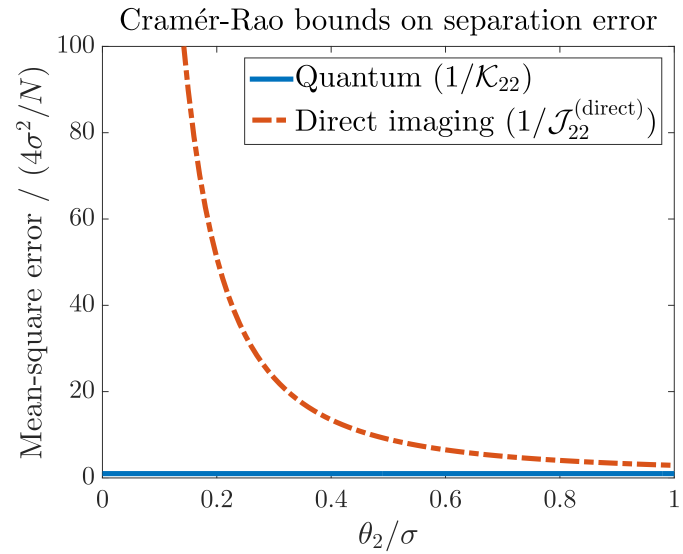
Quantum Metrology with Classical Sources

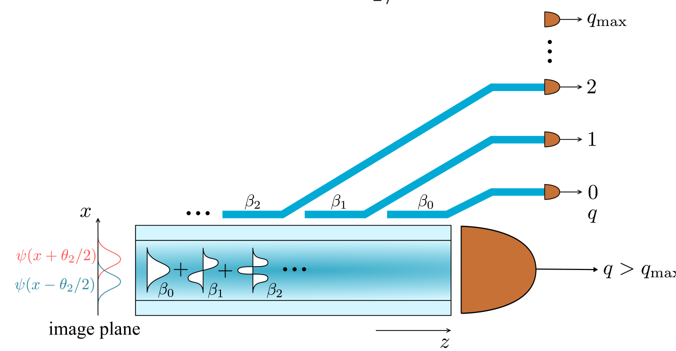
- **Thermal/fluorescent/laser sources**, **linear optics**, **photon counting**
- Compare with other QIP applications:
  - QKD (Bennett *et al.*)
  - Nonclassical-state metrology (Yuen, Caves)
  - Shor's algorithm
  - Quantum simulations (Feynman, Lloyd)
  - Boson sampling (Aaronson)
- More robust against **loss**
- **Rayleigh's criterion** is a huge deal for microscopy (physics, chemistry, biology, engineering), telescopy (astronomy), radar/lidar (military), etc.
- Current microscopy limited by **photon shot noise** in EMCCD (see, e.g., Pawley *ed.*, *Handbook of Biological Confocal Microscopy*).
- Ground telescopes/stellar interferometers limited by **atmospheric turbulence**, **space telescopes** are diffraction/shot-noise-limited.
- **Linear optics/photon-counting technology** is mature

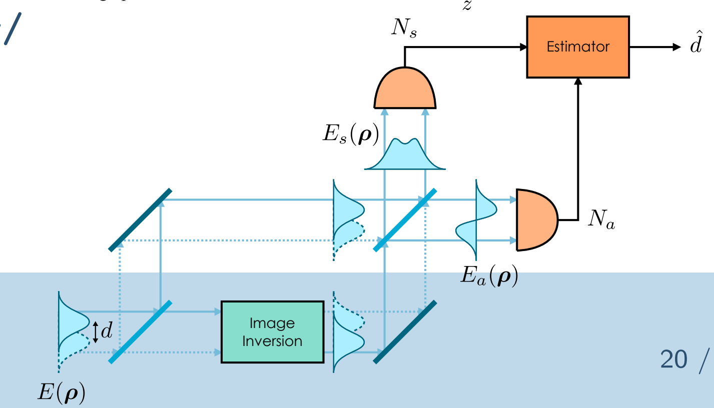
> Although any given scheme can be explained by a semiclassical model, quantum metrology remains a powerful tool for exploring the **ultimate performance with any measurement**.

Quantum Metrology Kills Rayleigh's Criterion

- Rayleigh's criterion is **irrelevant** to quantum bound.
- SPADE/SLIVER can achieve quantum bound via **linear optics** and **photon counting**.
    - Tsang, Nair, and Lu "Quantum theory of super-resolution for two incoherent optical point sources," arXiv:1511.00552.
    - Nair and Tsang, "Interferometric superlocalization for two incoherent optical point sources," Opt. Express **24**, 3684 (2016).
    - Tsang, Nair, and Lu, "Semiclassical Theory of Super-resolution for Two Incoherent Optical Point Sources," arXiv:1602.04655.
- **FAQ**: `https://sites.google.com/site/mankeitsang/news/rayleigh/faq`
- email: `mankei@nus.edu.sg`







$\epsilon \ll 1$ Approximation

- Chap. 9, Goodman, *Statistical Optics*:

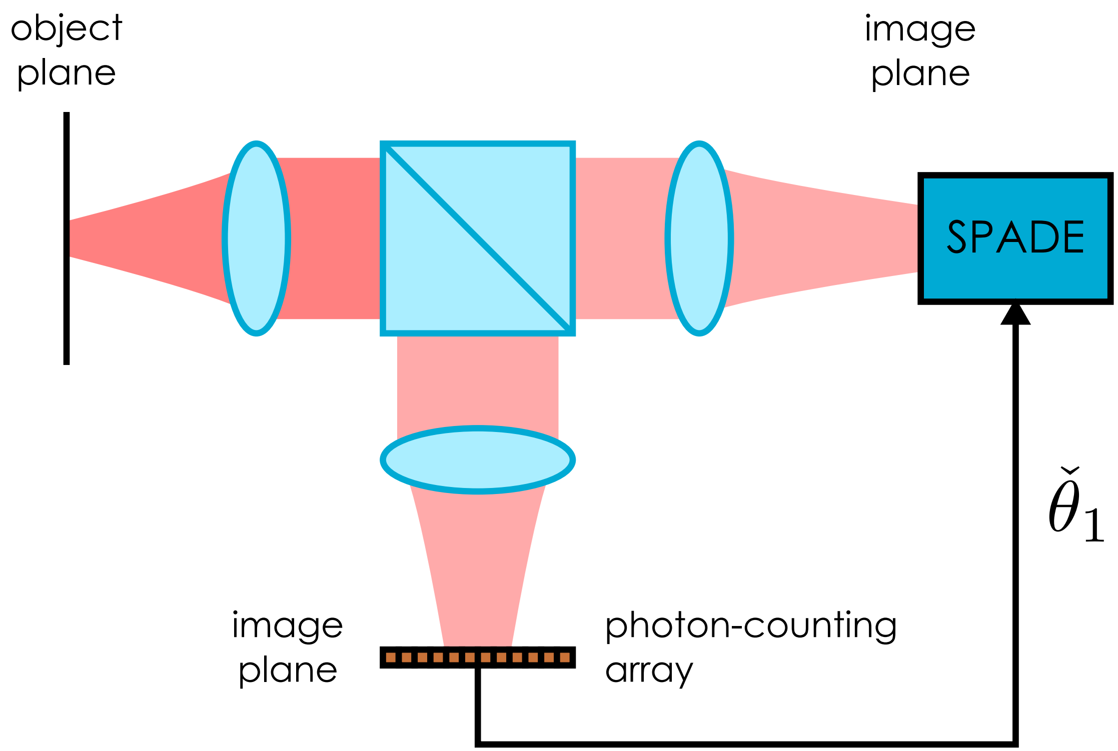
  "If the count degeneracy parameter is much less than 1, it is highly probable that there will be either **zero or one counts** *in each separate coherence interval of the incident classical wave. In such a case the classical intensity fluctuations have a* **negligible "bunching"** *effect on the photo-events, for (with high probability) the light is simply too weak to generate multiple events in a single coherence cell.*

- Zmuidzinas (https://pma.caltech.edu/content/jonas-zmuidzinas), JOSA A **20**, 218 (2003):

  "It is well established that the photon counts registered by the detectors in an optical instrument follow statistically independent **Poisson** *distributions, so that the fluctuations of the counts in different detectors are uncorrelated. To be more precise, this situation holds for the case of thermal emission (from the source, the atmosphere, the telescope, etc.) in which the mean photon occupation numbers of the modes incident on the detectors are low, $n \ll 1$. In the high occupancy limit, $n \gg 1$, photon bunching becomes important in that it changes the counting statistics and can introduce correlations among the detectors. We will discuss only the first case, $n \ll 1$,* **which applies to most astronomical observations at optical and infrared wavelengths**.*"*

- Hanbury Brown-Twiss (post-selects on two-photon coincidence): poor SNR, **obsolete for decades** in astronomy.
- See also Labeyrie *et al.*, *An Introduction to Optical Stellar Interferometry*, etc.
- Fluorescent particles: Pawley *ed.*, *Handbook of Biological Confocal Microscopy*, Ram, Ober, Ward (2006), etc., may have **antibunching**, but Poisson model is fine because of $\epsilon \ll 1$.

Unknown Centroid and Misalignment



Overhead to locate centroid:

$$\xi = \frac{|\theta_1 - \check{\theta}_1|}{\sigma} \sim \frac{1}{\sqrt{N_1}} \ll 1 \tag{19}$$